Elementary operator criterion of entanglement of quantum states

Jinchuan Hou

Institute of Mathematics, Department of Mathematics

Taiyuan University of Technology, Taiyuan 030024, P. R. of China

jinchuanhou@yahoo.com.cn, houjinchuan@tyut.edu.cn

1. Introduction

Positive linear maps and completely positive linear maps are found to be very important in quantum mechanics, quantum computation and quantum information. In fact they can be used to recognize entangled states, and every quantum channel is represented as a trace preserving completely positive linear map.

In quantum mechanics, a quantum system is associated with a separable complex Hilbert space H, i.e., the state space. A quantum state is described as a density operator $\rho \in \mathcal{T}(H) \subseteq \mathcal{B}(H)$ which is positive and has trace 1, where $\mathcal{B}(H)$ and $\mathcal{T}(H)$ denote the von Neumann algebras of all bounded linear operators and the trace-class of all operators T with $\|T\|_1 = \mathrm{Tr}((T^\dagger T)^{\frac{1}{2}}) < \infty$, respectively. ρ is a pure state if $\rho^2 = \rho$; ρ is a mixed state if $\rho^2 \neq \rho$. The state space H of a composite quantum system is the tensor product of the state spaces of the component quantum systems H_i, that is $H = H_1 \otimes H_2 \otimes \ldots \otimes H_k$. In this lecture we are mainly interested in bipartite systems, that is, the case $k = 2$. Let H and K be finite dimensional and let ρ be a state acting on $H \otimes K$. ρ is said to be separable if ρ can be written as

$$\rho = \sum_{i=1}^{k} p_i \rho_i \otimes \sigma_i,$$

where ρ_i and σ_i are states on H and K respectively, and p_i are positive numbers with $\sum_{i=1}^{k} p_i = 1$. Otherwise, ρ is said to be inseparable or entangled (ref. [2, 34]). For the infinite dimensional case, by Werner [46], a state ρ acting on $H \otimes K$ is called separable if it can be approximated in the trace norm by the states of the form

$$\sigma = \sum_{i=1}^{n} p_i \rho_i \otimes \sigma_i,$$

where ρ_i and σ_i are states on H and K respectively, and p_i are positive numbers with $\sum_{i=1}^{n} p_i = 1$. Otherwise, ρ is called an entangled state.

The quantum entangled states have been used as basic resources in quantum information processing and communication (see [3, 4, 15, 16, 34, 39]). Generally, to decide whether or not a state of composite quantum systems is entangled is one of the most challenging task of this field [34]. For the case of 2×2 or 2×3 systems, that is, for the case $\dim H = \dim K = 2$ or $\dim H = 2$, $\dim K = 3$, a state is separable if and only if it is a PPT (Positive Partial Transpose) state [22, 36]. But PPT is only a necessary condition for a state to be separable acting on Hilbert space of higher dimensions. There are PPT states that are entangled. It is known that PPT entangled states belong to the class of bound entangled states [23]. In [7], the realignment criterion for separability in finite dimensional systems was established, and was generalized to the infinite dimensional systems by [21]. The realignment criterion is a powerful criterion that is independent of the PPT criterion. However, there are still entangled states that can be recognized by neither the PPT criterion nor the realignment criterion. There are several other sufficient criteria for entanglement such as the reduction criterion and majorization criterion [6, 24, 25].

A most general approach to study the entanglement of quantum states in finite dimensional systems is based on the notion of entanglement witnesses (see [22]). A Hermitian operator W acting on $H \otimes K$ is said to be an entanglement witness (briefly, EW), if W is not positive and $\mathrm{Tr}(W\sigma) \geq 0$ holds for all separable states σ. Thus, ρ is entangled if and only if there exists an entanglement witness W such that $\mathrm{Tr}(W\rho) < 0$ [22]. This entanglement witness criterion is also valid for infinite dimensional systems. Clearly, constructing entanglement witnesses is a hard task. A recent result in [30] states that every entangled state in a bipartite (finite or infinite dimensional) system can be detected by a witness of the form $cI - F$, where c is a nonnegative number and F is a finite rank self-adjoint operator. A method of constructing entanglement witnesses of the form $I - F$ was also given in [30].

Another general approach to detect entanglement is based on positive maps. It is obvious that if ρ is a state on $H \otimes K$, then for every completely positive (briefly, CP) linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, the operator $(\Phi \otimes I)\rho \in \mathcal{B}(K \otimes K)$ is always positive; if ρ is separable, then for every positive linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, the operator $(\Phi \otimes I)\rho$ is always positive on $K \otimes K$ (or, for every positive linear map $\Phi : \mathcal{B}(K) \to \mathcal{B}(H)$, the operator $(I \otimes \Phi)\rho$ is always positive on $H \otimes H$). The converse of the last statement is also true. In [22], it was shown that

Horodeckis' Theorem. [22, Theorem 2] *Let H, K be finite dimensional complex Hilbert spaces and ρ be a state acting on $H \otimes K$. Then ρ is separable if and only if for any positive linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, the operator $(\Phi \otimes I)\rho$ is positive on $K \otimes K$.*

The positive map criterion and the witness criterion for entanglement are two of few known necessary and sufficient criteria. These two criteria are closely connected by the so-called the Jamiołkowski-Choi isomorphism [20, 22, 38, 40]. Recall that a positive map is said to be decomposable if it is the sum of a CP map and a map which is the transpose of some CP map. It is obvious that a decomposable positive map can not detect any PPT entangled states [32].

Let us consider the case that at least one of H and K is of infinite dimension. As every positive linear map η between von Neumann algebras is bounded and $\|\eta\| = \|\eta(I)\|$ (see [31, Exercise 10.5.10]), ρ is separable on $H\otimes K$ still implies that, for any completely bounded positive linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$, the operator $(\Phi\otimes I)\rho$ is positive on $K\otimes K$. The infinite-dimensional version of Horodeckis' Theorem above was obtained by Størmer [44]. Recall that a positive linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is said to be normal if it is weakly continuous on bounded sets, or equivalently, if it is ultra-weakly continuous (i.e., if $\{A_\alpha\}$ is a bounded net and there is $A\in\mathcal{B}(H)$ such that $\langle x|A_\alpha|y\rangle$ converges to $\langle y|A|x\rangle$ for any $|x\rangle\in H, |y\rangle\in K$, then $\langle x|\Phi(A_\alpha)|y\rangle$ converges to $\langle y|\Phi(A)|x\rangle$ for any $|x\rangle\in H, |y\rangle\in K$. ref. [17, pp.59]).

Størmer's Theorem. [44] *Let H, K be Hilbert spaces, ρ be a state acting on $H\otimes K$. Then ρ is separable if and only if for any normal positive linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$, the operator $(\Phi\otimes I)\rho$ is positive on $K\otimes K$.*

Thus, for a state ρ on $H\otimes K$, if there exists a normal positive map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ such that $(\Phi\otimes I)\rho$ is *not positive or unbounded*, then ρ is entangled. In this situation, Φ can never be completely positive. Therefore, to detect the inseparability of states, the key is to find the normal positive linear maps that are not completely positive. In the case that $\dim H=\dim K=n$, the transpose $A\mapsto A^T$ and the map $A\mapsto \mathrm{Tr}(A)I-A$ are well known positive maps that are not completely positive (briefly, NCP).

However, Størmer's Theorem is difficult to apply. To detect a state (1) we have to exhaust all normal positive linear maps; (2) the structure of the normal positive linear maps is not clear; (3) for a given entangled state, we do not know how to construct some non-completely positive normal positive linear maps that recognize the entanglement of this state.

Thus several natural questions rise. For instance, (1) is the set of completely bounded normal positive (CBNP) linear maps sufficient to determine the separability of any state? (2) are there any more small subsets of CBNP linear maps that are still enough to provide necessary and sufficient criteria of separability? (3) is there a tractable small subset of CBNP linear maps that is enough to provide a necessary and sufficient criterion of separability?

In this lecture note, we give a characterization of completely bounded normal positive linear maps, and show that *the set of all positive finite-rank elementary operators* is enough to provide a necessary and sufficient criterion of the separability of states in infinite dimensional systems. We also illustrate how to construct NCP positive finite-rank elementary operators and apply them to detect the entanglement of some states.

Positive linear maps have attracted much attention of physicists working in quantum information science in recent decades because of Horodeckis' positive map criterion. Great efforts have be payed to find as many as possible positive maps that are not CP, and then use them to detect some entangled states [1, 11, 12, 25], for finite dimensional systems.

Positive linear maps and completely positive linear maps are also important mathematical topics studied intensively in a general setting of C*-algebras by mathematicians. The completely positive linear maps can be understood quite well (Stinespring's theorem). However, the structure of positive linear maps is drastically nontrivial even for the finite dimensional case ([8]-[10], [35]).

Note that every linear map Φ from $\mathcal{B}(H)$ into $\mathcal{B}(K)$ is an elementary operator if both H and K are finite dimensional, that is, there exist operators $A_1, A_2, \ldots, A_k \in \mathcal{B}(H, K)$ and $B_1, B_2, \ldots, B_k \in \mathcal{B}(K, H)$, such that $\Phi(T) = \sum_{i=1}^{k} A_i T B_i$ for all $T \in \mathcal{B}(H)$. So, it is also basic important and interesting to find as many as possible characterizations of positive elementary operators and characterizations of completely positive elementary operators, and then, to apply them to get some criteria for the entanglement of states.

A characterization of positive elementary operators was obtained in [28] in terms of contractively locally linear combinations. This is the only known necessary and sufficient condition for an elementary operator to be positive. In this lecture we give a characterization of positive completely bounded normal maps between $\mathcal{B}(H)$ and $\mathcal{B}(K)$, which including all positive elementary operators. Consequently, we present some concrete representations of the completely bounded linear maps, positive completely bounded linear maps and completely positive linear maps between the trace-classes $\mathcal{T}(H)$ and $\mathcal{T}(K)$, which allow us to obtain a representation of quantum channels (operations) for infinite-dimensional systems. Apply these characterization of positive maps that are not CP, a necessary and a sufficient criterion, that is, the elementary operator criterion of separability is established. Finally, some positive elementary operators are constructed so that they are not completely positive, even indecomposable, and then used to recognize some entangled quantum states that cannot be detected by the PPT criterion and the realignment criterion.

Throughout, H and K are separable complex Hilbert spaces that may be of infinite dimension if no specific assumption is made, and $\langle\cdot|\cdot\rangle$ stands for the inner product in both of them. $\mathcal{B}(H, K)$ ($\mathcal{B}(H)$ when $K = H$) is the Banach space of all (bounded linear) operators from H into K. $A \in \mathcal{B}(H)$ is self-adjoint if $A = A^\dagger$ ($A^\dagger$ stands for the adjoint operator of A); and A is positive, denoted by $A \geq 0$, if A is self-adjoint with spectrum falling in the interval $[0, \infty)$ (or equivalently, $\langle\psi|A\psi\rangle \geq 0$ for all $|\psi\rangle \in H$). For any positive integer n, $H^{(n)}$ denotes the direct sum of n copies of H. It is clear that every operator $\mathbf{A} \in \mathcal{B}(H^{(n)}, K^{(m)})$ can be written in an $m \times n$ operator matrix $\mathbf{A} = (A_{ij})_{i,j}$ with $A_{ij} \in \mathcal{B}(H, K)$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Equivalently, $\mathcal{B}(H^{(n)}, K^{(m)})$ is often written as $\mathcal{B}(H, K) \otimes \mathcal{M}_{m\times n}(\mathbb{C})$. We will write $\mathbf{A}^{\mathrm{T}} = (A_{ij})^{\mathrm{T}}$ for the formal transpose matrix $(A_{ji})_{i,j}$ of $\mathbf{A}$, $\mathbf{A}^{\mathrm{t}} = (A_{ji}^{\mathrm{t}})_{i,j}$ for the usual transpose of $\mathbf{A}$, and denote by $A^{(n)}$ the operator matrix $(A_{ij}) \in \mathcal{B}(H^{(n)}, K^{(n)})$ with $A_{ii} = A$ and $A_{ij} = 0$ if $i \neq j$. If Φ is a linear map from $\mathcal{B}(H)$ into $\mathcal{B}(K)$, we can define a linear map $\Phi_n : \mathcal{B}(H^{(n)}) \to \mathcal{B}(K^{(n)})$ by $\Phi_n((A_{ij})) = (\Phi(A_{ij}))$. Recall that Φ is said to be positive (resp. hermitian-preserving) if $A \in \mathcal{B}(H)$ is positive (resp. self-adjoint) implies that $\Phi(A)$ is positive (resp. self-adjoint). If Φ_n is positive we say Φ is n-positive; if Φ_n is positive for every integer $n > 0$, we say that Φ is completely positive. Obviously, Φ is completely positive $\Rightarrow$ Φ is positive $\Rightarrow$ Φ is hermitian-preserving. Φ is said to be completely bounded if $\|\phi\|_{cb} = \sup_n \|\Phi_n\| < \infty$.

2. Characterizing positive completely bounded normal linear maps

In this section we give a characterization of positive completely bounded normal linear maps from $\mathcal{B}(H)$ into $\mathcal{B}(K)$.

Recall that a linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is called an elementary operator if there are two finite sequences $\{A_i\}_{i=1}^n \subset \mathcal{B}(H,K)$ and $\{B_i\}_{i=1}^n \subset \mathcal{B}(K,H)$ such that $\Phi(X) = \sum_{i=1}^n A_i X B_i$ for all $X \in \mathcal{B}(H)$; $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is called a generalized elementary operator if there exist sequences $\{A_i\}$ and $\{B_i\}$ satisfying $\|\sum_i A_i A_i^\dagger\| \|\sum_i B_i^\dagger B_i\| < \infty$ such that $\Phi(X) = \sum_i A_i X B_i$ for all X. Obviously, the generalized elementary operators are completely bounded and normal.

We first give a lemma.

Lemma 2.1. *Let H, K be separable complex Hilbert spaces and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a linear map. Then Φ is normal and completely bounded if and only if Φ is a generalized elementary operator.*

Proof. We need only check the "only if" part. Assume that the linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is completely bounded and normal. It follows that, $\Phi = \Phi_1 - \Phi_2 + i(\Phi_3 - \Phi_4)$ with Φ_i normal and completely positive by Wittstock's decomposition theorem (ref. [35]). As H and K are separable, by Stinespring's Theorem (ref. [35, 43]) and the structural theorem of normal $*$-homomorphisms of $\mathcal{B}(H)$ (ref. [17, pp.61]), for each $k = 1, 2, 3, 4$, there exist a countable cardinal number J_k, an operator $U_k \in \mathcal{B}(H^{(J_k)}, K)$ such that $\Phi_k(X) = U_k X^{(J_k)} U_k^\dagger$, where $H^{(J_k)}$ (resp. $X^{(J_k)}$) is the direct sum of J_k-copies of H (resp. of X). Therefore, there are sequences of operators $\{A_i\}_{i\leq J_1}, \{B_j\}_{j\leq J_2}, \{C_s\}_{s\leq J_3}, \{D_t\}_{t\leq J_4} \subset \mathcal{B}(H,K)$, such that

$$U_1 = \begin{pmatrix} A_1 & A_2 & \cdots & A_i & \cdots \end{pmatrix}$$

$$U_2 = \begin{pmatrix} B_1 & B_2 & \cdots & B_j & \cdots \end{pmatrix},$$

$$U_3 = \begin{pmatrix} C_1 & C_2 & \cdots & C_s & \cdots \end{pmatrix},$$

$$U_4 = \begin{pmatrix} D_1 & D_2 & \cdots & D_t & \cdots \end{pmatrix}$$

and

$$\Phi(X) = \sum_{i\leq J_1} A_i X A_i^\dagger - \sum_{j\leq J_2} B_j X B_j^\dagger + i \sum_{s\leq J_3} C_s X C_s^\dagger - i \sum_{t\leq J_4} D_t X D_t^\dagger$$

for every $X \in \mathcal{B}(H)$. Now it is clear that

$$\|\sum_{i\leq J_1} A_i A_i^\dagger + \sum_{j\leq J_2} B_i B_i^\dagger + \sum_{s\leq J_3} C_s C_s^\dagger + \sum_{\leq J_4} D_t D_t^\dagger\| \leq \sum_{k=1}^4 \|U_k\|^2 < \infty,$$

and so, Φ is a generalized elementary operator. □

By Lemma 2.1, *the question of characterizing positive completely bounded normal linear maps between $\mathcal{B}(H)$ and $\mathcal{B}(K)$ is equivalent to the question of characterizing positive generalized elementary operators.*

As a special class of generalized elementary operators, the global structures of hermitian-preserving and completely positive elementary operators are quite clear. In fact, for generalized elementary operators, by the proof of Lemma 2.1, we have the following result.

Corollary 2.2. *Let H, K be Hilbert spaces and Φ be a generalized elementary operator from $\mathcal{B}(H)$ into $\mathcal{B}(K)$. Then*

(i) *Φ is hermitian-preserving if and only if there are sequences $\{A_i\}, \{C_j\} \subset \mathcal{B}(H,K)$ with $\| \sum_{i=1}^{\infty} A_i A_i^\dagger \| < \infty$ and $\| \sum_{j=1}^{\infty} C_j C_j^\dagger \| < \infty$ such that*

$$\Phi(X) = \sum_{i=1}^{\infty} A_i X A_i^\dagger - \sum_{j=1}^{\infty} C_j X C_j^\dagger$$

for every $X \in \mathcal{B}(H)$;

(ii) *Φ is completely positive if and only if there exists a sequence $\{A_i\} \subset \mathcal{B}(H,K)$ with $\| \sum_{i=1}^{\infty} A_i A_i^\dagger \| < \infty$ such that*

$$\Phi(X) = \sum_{i=1}^{\infty} A_i X A_i^\dagger$$

for every $X \in \mathcal{B}(H)$.

If both H and K are finite-dimensional, Corollary 2.2 (i) and (ii) were established by DePillis [14] and Choi [8], respectively. For the elementary operator case, see [26] and [33].

For a sequence $\mathbf{A} = (\; A_1 \quad A_2 \quad \cdots \quad A_i \quad \cdots \;)$, we will denote by $\mathbf{A}^T$ the formal transpose of $\mathbf{A}$ and $\mathbf{A}^\dagger$ the usual adjoint operator of $\mathbf{A}$, that is

$$\mathbf{A}^T = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_i \\ \vdots \end{pmatrix} \quad \text{and} \quad \mathbf{A}^\dagger = \begin{pmatrix} A_1^\dagger \\ A_2^\dagger \\ \vdots \\ A_i^\dagger \\ \vdots \end{pmatrix}.$$

We will also denote by $\mathcal{B}_1(H,K)$ the closed unit ball of $\mathcal{B}(H,K)$.

The next lemma is the key lemma which is a generalization of [28, Lemma 2.2], where more conditions $\| \sum_{i=1}^{\infty} A_i^\dagger A_i \| < \infty$ and $\| \sum_{j=1}^{\infty} C_j^\dagger C_j \| < \infty$ are assumed. Note that, the conditions $\| \sum_{i=1}^{\infty} A_i A_i^\dagger \| < \infty$ and $\| \sum_{i=1}^{\infty} A_i^\dagger A_i \| < \infty$ are not equivalent in general. For instance, let $H = \oplus_{i=1}^{\infty} H_i$ with each H_i is of infinite dimension. Let $V_i \in \mathcal{B}(H)$ be the isometry with range H_i. Then $V_i^\dagger V_i = I$ and $V_i V_i^\dagger = P_i$, where P_i is the projection from H onto H_i. Thus $\| \sum_{i=1}^{\infty} V_i V_i^\dagger \| = \| \sum_{i=1}^{\infty} P_i \| = \|I\| = 1$ as $P_i P_j = 0$ whenever $i \neq j$, but $\| \sum_{i=1}^{\infty} V_i^\dagger V_i \| = \infty$.

Lemma 2.3. *Let $\{A_i\}_{i=1}^{\infty}$ and $\{C_j\}_{j=1}^{\infty} \subset \mathcal{B}(H,K)$ with $\| \sum_{i=1}^{\infty} A_i A_i^\dagger \| < \infty$ and $\| \sum_{j=1}^{\infty} C_j C_j^\dagger \| < \infty$. Then the following statements are equivalent:*

(i) *$\sum_{i=1}^{\infty} A_i P A_i^\dagger \geq \sum_{j=1}^{\infty} C_j P C_j^\dagger$ for all positive operators $P \in \mathcal{B}(H)$.*

(ii) *$\sum_{i=1}^{\infty} A_i P A_i^\dagger \geq \sum_{j=1}^{\infty} C_j P C_j^\dagger$ for all rank-one projections $P \in \mathcal{B}(H)$.*

(iii) *There exists a map $\Omega : H \to \mathcal{B}_1(l_2)$ such that*

$$\mathbf{C}^T |\psi\rangle = \Omega(|\psi\rangle) \mathbf{A}^T |\psi\rangle \quad \textit{for every} \quad |\psi\rangle \in H.$$

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii). Given any $|\psi\rangle \in H$ with $\||\psi\rangle\| = 1$, $P = |\psi\rangle\langle\psi|$ is a rank-one projection. From (ii) we have

$$\sum_{i=1}^{\infty} A_i P A_i^\dagger \geq \sum_{j=1}^{\infty} C_j P C_j^\dagger. \tag{2.1}$$

Let

$$\mathbf{T} = \begin{pmatrix} A_1P & \cdots & A_iP & \cdots \\ 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} C_1P & \cdots & C_jP & \cdots \\ 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

be operators from $H^{(\infty)}$ into $K^{(\infty)}$. The inequality (2.1) implies that $\mathbf{TT}^\dagger \geq \mathbf{SS}^\dagger$. So by [18], there exists a unique contractive operator $\mathbf{X} = (X_{ij}) \in \mathcal{B}(H^{(\infty)})$ such that $\ker \mathbf{X}^\dagger \supseteq \ker \mathbf{T}$ and $\mathbf{S} = \mathbf{TX}$. Since

$$\begin{pmatrix} 0 & \cdots & 0 & |\phi_i\rangle & 0 & \cdots \end{pmatrix}^T \in \ker \mathbf{T}$$

for each $|\phi_i\rangle \in \ker A_iP$, we have $X_{ij}^\dagger|\phi_i\rangle = 0$ for all $j = 1, 2, \ldots$. Hence, $\ker X_{ij}^\dagger \supseteq \ker A_iP$ for all i and j. It follows that X_{ij} are operators of rank at most one and there exist vectors $|\phi_{ij}\rangle \in H$ such that $X_{ij} = |\psi\rangle\langle\phi_{ij}|$ for all $i, j = 1, 2, \ldots$. Now $\mathbf{S} = \mathbf{TX}$ leads to

$$\begin{aligned} & \begin{pmatrix} C_1|\psi\rangle & \cdots & C_j|\psi\rangle & \cdots \end{pmatrix}^T \\ = & \begin{pmatrix} C_1P|\psi\rangle & \cdots & C_jP|\psi\rangle & \cdots \end{pmatrix}^T \\ = & \begin{pmatrix} \sum_{i=1}^{\infty} A_iPX_{i1}|\psi\rangle & \cdots & \sum_{i=1}^{\infty} A_iPX_{ij}|\psi\rangle & \cdots \end{pmatrix}^T \\ = & \begin{pmatrix} \sum_{i=1}^{\infty} \langle\phi_{i1}|\psi\rangle A_i|\psi\rangle & \cdots & \sum_{i=1}^{\infty} \langle\phi_{ij}|\psi\rangle A_i|\psi\rangle & \cdots \end{pmatrix}^T. \end{aligned}$$

Denote $\omega_{ji} = \langle\phi_{ij}|\psi\rangle$ and let $\Omega(|\psi\rangle) = (\omega_{ji}(|\psi\rangle))_{j,i}$. Then we have

$$\begin{aligned} \mathbf{C}^T|\psi\rangle = & \begin{pmatrix} C_1|\psi\rangle & \cdots & C_j|\psi\rangle & \cdots \end{pmatrix}^T \\ = & \Omega(|\psi\rangle)\begin{pmatrix} A_1|\psi\rangle & \cdots & A_i|\psi\rangle & \cdots \end{pmatrix}^T = \Omega(|\psi\rangle)\mathbf{T}^T|\psi\rangle. \end{aligned}$$

Moreover, since $X_{ij}P = \langle\phi_{ij}|\psi\rangle P = \omega_{ji}(|\psi\rangle)P$, by regarding $\Omega(|\psi\rangle)$ as an operator from l_2 into itself, we get

$$\|\Omega(|\psi\rangle)\| = \|\Omega(|\psi\rangle) \otimes P\| = \|\mathbf{X}P^{(\infty)}\| \leq \|\mathbf{X}\| \leq 1.$$

Therefore, (ii) holds implies that (iii) holds.

(iii)⇒(ii). Assume (iii). For any unit vector $|\psi\rangle \in H$, denote $P = |\psi\rangle\langle\psi|$ and the contractive matrix $\Omega(|\psi\rangle) = \Omega = (\omega_{ji})$. As $\mathbf{C}^T|\psi\rangle = \Omega(|\psi\rangle)\mathbf{A}^T|\psi\rangle$, we have $C_j|\psi\rangle = \sum_{j=1}^{\infty} \omega_{ji}A_i|\psi\rangle$ for each i. Thus,

$$\begin{aligned} \mathbf{C}P = & \begin{pmatrix} C_1P & C_2P & \cdots & C_jP & \cdots \end{pmatrix} \\ = & \begin{pmatrix} \sum_{i=1}^{\infty} \omega_{1i}A_iP & \sum_{i=1}^{\infty} \omega_{2i}A_iP & \cdots & \sum_{i=1}^{\infty} \omega_{ji}A_iP & \cdots \end{pmatrix} \\ = & \begin{pmatrix} A_1P & A_2P & \cdots & A_iP & \cdots \end{pmatrix}\Omega^T \\ = & \begin{pmatrix} A_1P & A_2P & \cdots & A_iP & \cdots \end{pmatrix}(w_{ji}I)^T = \mathbf{A}P(w_{ji}I)^T. \end{aligned}$$

It follows that

$$\sum_{j=1}^{\infty} C_j P C_j^\dagger = \mathbf{C}P\mathbf{C}^\dagger = \mathbf{A}P(\omega_{ji}I)^T((\omega_{ji}I)^T)^\dagger P\mathbf{A}^\dagger \leq \mathbf{A}P\mathbf{A}^\dagger = \sum_{i=1}^{\infty} A_i P A_i^\dagger$$

because of $0 \leq (\omega_{ji}I)^T((\omega_{ji}I)^T)^\dagger \leq I$.

(ii)⇒(i). Let $\Delta(X) = \sum_{i=1}^{\infty} A_i X A_i^\dagger - \sum_{j=1}^{\infty} C_j X C_j^\dagger = \mathbf{A}X^{(\infty)}\mathbf{A}^\dagger - \mathbf{C}X^{(\infty)}\mathbf{C}^\dagger$ for each $X \in \mathcal{B}(H)$. Since $\|\mathbf{A}\| = \|\mathbf{A}\mathbf{A}^\dagger\|^{\frac{1}{2}} = (\|\sum_{i=1}^{\infty} A_i A_i^\dagger\|)^{\frac{1}{2}} < \infty$ and $\|\mathbf{C}\| = \|\mathbf{C}\mathbf{C}^\dagger\|^{\frac{1}{2}} = (\|\sum_{j=1}^{\infty} C_j C_j^\dagger\|)^{\frac{1}{2}} < \infty$, we see that Δ is normal. The condition (ii) implies that $\Delta(P)$ is positive for every finite rank positive operator P. For any positive operator $X \in \mathcal{B}(H)$, by spectral theorem, there exists a net P_λ of finite-rank positive operators such that $\|P_\lambda\| \leq \|X\|$ and $\text{wk}-\lim_\lambda P_\lambda = X$. Hence $\Delta(X) = \text{wk}-\lim_\lambda \Delta(P_\lambda)$ is positive and (i) is true. □

Lemma 2.4. *Let H, K be complex Hilbert spaces and $\{A_i\}_{i=1}^{\infty}, \{C_j\}_{j=1}^{\infty} \subset \mathcal{B}(H,K)$ with $\| \sum_{i=1}^{\infty} A_i A_i^\dagger \| < \infty$ and $\| \sum_{j=1}^{\infty} C_j C_j^\dagger \| < \infty$. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a linear map defined by*

$$\Phi(X) = \sum_{i=1}^{\infty} A_i X A_i^\dagger - \sum_{j=1}^{\infty} C_j X C_j^\dagger$$

for every $X \in \mathcal{B}(H)$. Then

(i) *Φ is positive if and only if there exists a map $\Omega : |\psi\rangle \in H \mapsto \Omega(|\psi\rangle) = (\omega_{ji}(|\psi\rangle))_{j,i} \in \mathcal{B}_1(l_2)$ such that*

$$\mathbf{C}^T|\psi\rangle = \Omega(|\psi\rangle)\mathbf{A}^T|\psi\rangle$$

for every $|\psi\rangle \in H$.

(ii) *Φ is completely positive if and only if there exists a contractive matrix $\Omega = (\omega_{ji})_{j,i} \in \mathcal{B}(l_2)$ such that*

$$\mathbf{C}^T = \Omega\mathbf{A}^T,$$

and in turn, if and only if there exists a sequence $\{D_i\}_{i=1}^{\infty} \subset \mathcal{B}(H,K)$ such that

$$\Phi(X) = \sum_{i=1}^{\infty} D_i X D_i^\dagger.$$

holds for all $X \in \mathcal{B}(H)$.

Here $\mathbf{A} = \begin{pmatrix} A_1 & A_2 & \cdots & A_i & \cdots \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} C_1 & C_2 & \cdots & C_j & \cdots \end{pmatrix}$.

Proof. By Lemma 2.3, (i) is true. By Corollary 2.2, Φ is completely positive if and only if there exists a sequence $\{D_i\}_{i=1}^{\infty} \subset \mathcal{B}(H,K)$ such that

$$\Phi(X) = \sum_{i=1}^{\infty} D_i X D_i^\dagger$$

for every $X \in \mathcal{B}(H)$. So, to complete the proof of Lemma 2.4, we need to show that Φ is completely positive if and only if there exists a contractive matrix $\Omega = (\omega_{ji})_{j,i} \in \mathcal{B}(l_2)$ such that

$$\mathbf{C}^T = \Omega\mathbf{A}^T. \tag{2.2}$$

Assume that Φ is completely positive. Then, for any positive integer n, we have

$$\sum_{i=1}^{\infty} A_i^{(n)} P A_i^{(n)\dagger} \geq \sum_{j=1}^{\infty} C_j^{(n)} P C_j^{(n)\dagger} \tag{2.3}$$

holds for all positive operators $P \in \mathcal{B}(H^{(n)})$. Let

$$\mathcal{B} = \{\Gamma \mathbf{A}^T : \Gamma = (\gamma_{ji})_{j,i} \in \mathcal{B}_1(l_2)\},$$

where $\mathcal{B}_1(l_2)$ stands for the closed unit ball of $\mathcal{B}(l_2)$. It is clear that $\mathcal{B}$ is closed in the strong operator topology in $\mathcal{B}(H, K^{(\infty)})$. Given $\varepsilon > 0$. For any $|x_1\rangle, \ldots, |x_n\rangle \in H$, let $|\mathbf{x}\rangle = (\ |x_1\rangle \ \cdots \ |x_n\rangle\ ) \in H^{(n)}$. It follows from Lemma 2.3 that there exists $\Omega(|\mathbf{x}\rangle) = (\omega_{ji}(|\mathbf{x}\rangle)) \in \mathcal{B}_1(l_2)$ such that

$$\Omega(|\mathbf{x}\rangle)\mathbf{A}^{(n)^T}|\mathbf{x}\rangle = \mathbf{C}^{(n)^T}|\mathbf{x}\rangle.$$

Therefore,

$$\Omega(|\mathbf{x}\rangle)\mathbf{A}^T|x_k\rangle = \mathbf{C}^T|x_k\rangle$$

holds for every $k = 1, 2, \ldots, n$. Thus

$$\Omega(|\mathbf{x}\rangle)\mathbf{A}^T \in \{\mathbf{X} \in \mathcal{B}(H, K^{(\infty)}) : \|\mathbf{X}|x_k\rangle - \mathbf{C}^T|x_k\rangle\| < \varepsilon \text{ for } k = 1, 2, \ldots, n\}.$$

However, this means that every strong neighborhood of $\mathbf{C}^T$ has a nonempty intersection with $\mathcal{B}$ and hence, $\mathbf{C}^T \in \mathcal{B}$. So, there exists an $\Omega \in \mathcal{B}_1(l_2)$ such that $\mathbf{C}^T = \Omega\mathbf{A}^T$.

Conversely, assume that Eq.(2.2) holds. Then, for any positive integer n we have $\mathbf{C}^{(n)^T} = \Omega^{(n)}\mathbf{A}^{(n)^T}$. By Lemma 2.3 again we see that Eq.(2.3) holds true and hence Φ is completely positive. □

Combining Lemma 2.1 and Lemma 2.4, one gets the main result of this section immediately.

Theorem 2.5. *Let H, K be separable complex Hilbert spaces and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a completely bounded normal linear map. Then*

(1) Φ *is positive if and only if there exist $\{A_i\}_{i=1}^{\infty}, \{C_j\}_{j=1}^{\infty} \subset \mathcal{B}(H, K)$ with $\| \sum_{i=1}^{\infty} A_i A_i^\dagger \| < \infty$ and $\| \sum_{j=1}^{\infty} C_j C_j^\dagger \| < \infty$, and a map $\Omega : |\psi\rangle \in H \mapsto \Omega(|\psi\rangle) = (\omega_{ji}(|\psi\rangle))_{j,i} \in \mathcal{B}_1(l_2)$ satisfying*

$$\mathbf{C}^T|\psi\rangle = \Omega(|\psi\rangle)\mathbf{A}^T|\psi\rangle$$

for every $|\psi\rangle \in H$, such that

$$\Phi(X) = \sum_{i=1}^{\infty} A_i X A_i^\dagger - \sum_{j=1}^{\infty} C_j X C_j^\dagger$$

holds for every $X \in \mathcal{B}(H)$.

(2) Φ *is completely positive if and only if there exists a sequence $\{D_i\}_{i=1}^{\infty} \subset \mathcal{B}(H, K)$ with $\| \sum_{i=1}^{\infty} D_i D_i^\dagger \| < \infty$ such that*

$$\Phi(X) = \sum_{i=1}^{\infty} D_i X D_i^\dagger.$$

holds for all $X \in \mathcal{B}(H)$.

Here $\mathbf{A} = (\ A_1 \ \ A_2 \ \cdots \ A_i \ \cdots\ )$ and $\mathbf{C} = (\ C_1 \ \ C_2 \ \cdots \ C_j \ \cdots\ )$.

What does Theorem 2.5 mean? To understand Theorem 2.5 better, let us recall some notions from [28]. Let $l,\ k \in \mathbb{N}$ (the set of all natural numbers), and let $A_1, \cdots, A_k$, and $C_1, \cdots, C_l \in \mathcal{B}(H,\ K)$. If, for each $|\psi\rangle \in H$, there exists an $l \times k$ complex matrix $(\alpha_{ij}(|\psi\rangle))$ (depending on $|\psi\rangle$) such that

$$C_i|\psi\rangle = \sum_{j=1}^{k} \alpha_{ij}(|\psi\rangle)A_j|\psi\rangle, \qquad i = 1, 2, \cdots, l,$$

we say that $\{C_1, \cdots, C_l\}$ is a locally linear combination of $\{A_1, \cdots, A_k\}$, $(\alpha_{ij}(|\psi\rangle))$ is called a *local coefficient matrix* at $|\psi\rangle$. Furthermore, if a local coefficient matrix $(\alpha_{ij}(|\psi\rangle))$ can be chosen for every $|\psi\rangle \in H^{(n)}$ so that the operator norm $\|(\alpha_{ij}(|\psi\rangle))\| \leq 1$, we say that $\{C_1, \cdots, C_l\}$ is a *contractive locally linear combination* of $\{A_1, \cdots, A_k\}$; if there is a matrix (α_{ij}) with $\|(\alpha_{ij})\| \leq 1$ such that $C_i = \sum_{j=1}^{k} \alpha_{ij}A_j$ for all i, we say that $\{C_1, \cdots, C_l\}$ is a *contractive linear combination* of $\{A_1, \cdots, A_k\}$ with coefficient matrix (α_{ij}). These notions can be generalized to the case that there are infinite many A_ks or C_ks. For instance, if, for every $|\psi\rangle \in H$, there are scalars $\alpha_k(|\psi\rangle)$ such that $C|\psi\rangle = \sum_{k=1}^{\infty} \alpha_k(|\psi\rangle)A_k|\psi\rangle$ and $\sum_{k=1}^{\infty} |\alpha_k(|\psi\rangle)|^2 \leq 1$, we will say that C is a *generalized contractive locally linear combination* of $\{A_k\}_{k=1}^{\infty}$.

Thus Theorem 2.5 may be restated as follows: A completely bounded normal linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive but not completely positive if and only if it has the form $\Phi(X) = \sum_{i=1}^{\infty} A_iXA_i^\dagger - \sum_{j=1}^{\infty} C_jXC_j^\dagger$ for all X, where $\{C_j\}$ is a generalized contractive locally linear combination of $\{A_i\}$ but $\{C_j\}$ is not a generalized contractive linear combination of $\{A_i\}$. This characterization is much helpful in some sense to understand the differences of completely positive normal linear maps, positive completely bounded normal linear maps and hermitian completely bounded normal linear maps.

By Theorem 2.5, one gets immediately a global structure theorem for positive elementary operators in terms of locally linear combination that was established in [28]. For $\mathcal{L} \subset \mathcal{B}(H, K)$, we'll denote by $[\mathcal{L}]$ the linear span of $\mathcal{L}$.

Corollary 2.6. *Let $\Phi = \sum_{i=1}^{n} A_i(\cdot)B_i$ be an elementary operator from $\mathcal{B}(H)$ into $\mathcal{B}(K)$. Then Φ is positive if and only if there exist $C_1, \cdots, C_k$ and $D_1, \cdots, D_l$ in $[A_1, \cdots, A_n]$ with $k + l \leq n$ such that $(D_1, \cdots, D_l)$ is a contractive locally linear combination of $(C_1, \cdots, C_k)$ and*

$$\Phi = \sum_{i=1}^{k} C_i(\cdot)C_i^\dagger - \sum_{j=1}^{l} D_j(\cdot)D_j^\dagger. \tag{2.4}$$

Furthermore, Φ in Eq.(2.4) is completely positive if and only if $(D_1, \cdots, D_l)$ is a linear combination of $(C_1, \cdots, C_k)$ with a contractive coefficient matrix, and in turn, if and only if there exist $E_1, E_2, \ldots, E_r$ with $r \leq k$ such that

$$\Phi = \sum_{i=1}^{r} E_i(\cdot)E_i^\dagger.$$

In fact, a characterization of k-positive elementary operators is given in [28]. By the same spirit, we can get also a characterization of k-positive completely bounded normal linear maps by applying Theorem 2.5. However, positive linear maps are more powerful than the 2-positive linear maps if we use them to detect entanglement.

Since every linear map between matrix algebras is an elementary operator, by Corollary 2.6 we get a characterization of positive maps that is not CP for finite dimensional case.

Corollary 2.7. *Let H and K be finite dimensional complex Hilbert spaces and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a linear map. Then Φ is positive but not completely positive if and only if there exist $C_1, \cdots, C_k, D_1, \cdots, D_l \in \mathcal{B}(H, K)$ such that $\Phi(X) = \sum_{i=1}^{k} C_i X C_i^\dagger - \sum_{j=1}^{l} D_j X D_j^\dagger$ for all $X \in \mathcal{B}(H)$, and $\{D_j\}_{j=1}^{l}$ is a contractive locally linear combination but not a contractive linear combination of $\{C_i\}_{i=1}^{k}$.*

It is interesting to observe from the discussion above that, for elementary operators, the question when positivity ensures complete positivity may be reduced to the question when contractive locally linear combination implies linear combination. This connection allows us to look more deeply into the relationship and the difference between positivity and complete positivity, and obtain some simple criteria to check whether a positive elementary operator is completely positive or not. This is important especially when we construct positive maps and apply them to recognize entanglement.

If $\mathcal{L} \subset \mathcal{B}(H, K)$, we will denote by $\mathcal{L}_F$ the subset of all finite-rank operators in $\mathcal{L}$.

The Corollaries 2.8 and 2.9 below can be found in [28]. We list them here for completeness and for reader's convenience.

Corollary 2.8. *Assume that $\Phi = \sum_{i=1}^{k} A_i(\cdot)A_i^\dagger - \sum_{j=1}^{l} B_j(\cdot)B_j^\dagger : \mathcal{B}(H) \to \mathcal{B}(K)$ is a positive elementary operator. If any one of the following conditions holds, then Φ is completely positive:*

(i) $k \leq 2$.

(ii) $\dim[A_1, \cdots, A_k]_F \leq 2$.

(iii) *There exists a vector $|\psi\rangle \in H$ such that $\{|A_i\psi\rangle\}_{i=1}^{k}$ is linearly independent.*

(iv) Φ *is $[\frac{k+1}{2}]$-positive, where $[t]$ stands for the integer part of real number t.*

Corollary 2.9. *Assume that $\Phi = \sum_{i=1}^{k} A_i(\cdot)A_i^\dagger - \sum_{j=1}^{l} B_j(\cdot)B_j^\dagger : \mathcal{B}(H) \to \mathcal{B}(K)$ is a positive elementary operator. If Φ is not completely positive, then*

(i) $k \geq 3$;

(ii) $\dim[A_1, \cdots, A_k]_F \geq 3$;

(iii) *every B_j, $j = 1, 2, \ldots, l$, is a finite-rank perturbation of some combination of $\{A_i\}_{i=1}^{k}$;*

(iv) $\Phi_{[\frac{k+1}{2}]}$ *is not positive.*

Corollary 2.10. *Assume that $\Phi = \sum_{i=1}^{k} A_i(\cdot)A_i^\dagger - \sum_{j=1}^{l} B_j(\cdot)B_j^\dagger : \mathcal{B}(H) \to \mathcal{B}(K)$ is an elementary operator. If there exists some j such that B_j is not a contractive linear combination of $\{A_i\}_{i=1}^{k}$, then Φ is not completely positive.*

The following result reveals that the non-complete positivity of a positive elementary operator is essentially determined by its behavior on finite-dimensional subspaces. So, *to construct a NCP positive elementary operator, it is enough to consider the question in finite-dimensional cases.*

Theorem 2.11. *Assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a positive elementary operator. Then Φ is NCP if and only if there exist finite-rank projections P and Q acting on H and K, respectively, such that the positive elementary operator $\Delta : \mathcal{B}(PH) \to \mathcal{B}(QK)$ defined by $\Delta(X) = Q\Phi(PXP)Q|_{QK}$ is non-completely positive. In addition, P and Q may be taken so that $\Delta' : \mathcal{B}(\ker P) \to \mathcal{B}(\ker Q)$ defined by $\Delta'(Y) = (I-Q)\Phi(((I-P)Y(I-P))(I-Q)|_{\ker Q}$ is completely positive.*

Proof. Clearly, if $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a positive linear map and $P \in \mathcal{B}(H)$, $Q \in \mathcal{B}(K)$ are projections, then $\Delta : \mathcal{B}(PH) \to \mathcal{B}(QK)$ defined by $\Delta(X) = Q\Phi(PXP)Q$ is positive and Δ is NCP implies that Φ is NCP.

Assume that Φ is a positive elementary operator, writing $\Phi = \sum_{i=1}^k A_i(\cdot)A_i^\dagger - \sum_{j=1}^l B_j(\cdot)B_j^\dagger$ with $\{A_1, \ldots, A_k, B_1, \ldots, B_l\}$ linearly independent. By Corollary 2.9 (ii)-(iii), if Φ is NCP, then the linear subspace spanned by $\{A_i\}_{i=1}^k$ has many finite rank operators and there exists $C_j \in [A_1, A_2, \ldots, A_k]$ and finite rank operators $F_j \notin [A_1, \ldots, A_k]$ such that $B_j = C_j + F_j$. Let P_0 be the projection with range the finite dimensional linear subspace spanned by all the ranges of $\{E^\dagger : E \in [A_1, \ldots, A_k]_{\mathcal{F}}\}$ and the ranges of $\{F_j^\dagger\}_{j=1}^l$; and Q_0 the projection with range the finite dimensional linear subspace spanned by all the ranges of $\{E : E \in [A_1, \ldots, A_k]_{\mathcal{F}}\}$ and the ranges of $\{F_j\}_{j=1}^l$. It is easily checked that there exist some finite rank projections $P \geq P_0$ and $Q \geq Q_0$ such that $QB_jP \notin [QA_1P, \ldots, QA_kP]$ since $B_j \notin [A_1, \ldots, A_k]$. Pick such P and Q. Let $S_i = QA_i|_{PH}$, $i = 1, 2, \ldots, k$, and $T_j = QB_j|_{PH}$, $j = 1, 2, \ldots, l$. Let $\Delta : \mathcal{B}(PH) \to \mathcal{B}(QK)$ be the map defined by $\Delta(X) = \sum_{i=1}^k S_iXS_i^* - \sum_{j=1}^l T_jXT_j^* = Q\Phi(PXP)Q|_{QK}$. Then Δ is positive. By the choice of P and Q, T_j is not in $[S_1, \ldots, S_k]$ for some j. Hence, Δ is not completely positive by Corollary 2.9. Since $[(I-Q)A_1(I-P), \ldots, (I-Q)A_k(I-P)]_{\mathcal{F}} = \{0\}$, by Corollary 2.8, Δ' is completely positive. $\square$

To conclude this section, we give a simple example illustrating that how to use the results in this section to judge whether or not a map is positive, completely positive.

Example 2.12. Assume that $\dim H = n$ and $\{|i\rangle\}_{i=1}^n$ is an orthonormal basis. Denote $E_{ij} = |i\rangle\langle j|$. For a given positive number t, let $\Delta_t : \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear map defined by

$$\Delta_t(X) = t\sum_{i=1}^n E_{ii}XE_{ii} - X$$

for any $X \in \mathcal{B}(H)$. Then Δ_t is positive if and only if it is completely positive, and in turn, if and only if $t \geq n$.

In fact, let $A_i = \sqrt{t}E_{ii}$, then $\Delta_t(X) = \sum_{i=1}^n A_iXA_i^\dagger - IXI^\dagger$. It is clear that I is a linear combination of $A_1, \cdots, A_n$, i.e., $I = \sum_{i=1}^n \frac{1}{\sqrt{t}}A_i$. Then the sum of the square of the coefficients is $\sum_i(\frac{1}{\sqrt{t}})^2 = \frac{n}{t}$, and hence Δ_t is completely positive if and only if $t \geq n$ by Corollary 2.6. If $t < n$, then it is obvious that I is not a contractive locally linear combination of $A_1, \cdots, A_n$, and hence Δ_t is not positive.

3. Quantum channels for infinite dimensional systems

It is known that, for finite-dimensional quantum systems, a quantum channel (operation) $\mathcal{E}$ is a trace-preserving (trace-nonincreasing) completely positive linear map between associated matrix algebras and vice versa. Thus, by a result due to Choi [8], $\mathcal{E}$ is an elementary operator of the form $\mathcal{E}(\cdot)=\sum_{i=1}^n A_i(\cdot)A_i^\dagger$, where $\sum_{i=1}^n A_i^\dagger A_i = I$ ($\sum_{i=1}^n A_i^\dagger A_i \leq I$).

The dynamics of a closed quantum system are described by a unitary transform. A natural way to describe the dynamics of an open infinite dimensional quantum system (principal system) on H as a subsystem of a closed quantum system on $H\otimes H_{\rm env}$, composited by the principal system and an environment system. Let $\mathcal{E}$ be a channel on the principal system. Fix a state $\rho_{\rm env}\in\mathcal{S}(H_{\rm env})$; then there exists a unitary operator U acting on $H\otimes H_{\rm env}$ such that

$$\mathcal{E}(\rho)=\mathrm{Tr}_{\rm env}[U(\rho\otimes\rho_{\rm env})U^\dagger]. \tag{3.1}$$

Thus a quantum channel for infinite dimensional system is still a trace-preserving completely positive linear map between the trace-class operators. This raises the question of characterizing completely positive linear maps between trace-classes.

Using the discussion in Section 2, one can characterize the completely bounded linear maps, positive completely bounded linear maps and completely positive linear maps between the trace-classes. This allow us to obtain a similar representation of quantum operations for infinite-dimensional systems. Firstly we recall some notions. For $A\in\mathcal{B}(H)$, denote $|A|=(A^\dagger A)^{\frac12}$. Recall that the trace class $\mathcal{T}(H)=\{T : \|T\|_1=\mathrm{Tr}(|T|)<\infty\}$, which is an ideal of $\mathcal{B}(H)$. Furthermore, $\mathcal{T}(H)$ is a Banach space with the trace norm $\|\cdot\|_1$. The dual space of $\mathcal{T}(H)$ is $\mathcal{T}(H)^*=\mathcal{B}(H)$ and every bounded linear functional is of the form $T\mapsto\mathrm{Tr}(AT)$, where $A\in\mathcal{B}(H)$.

Lemma 3.1. *Let H, K be separable complex Hilbert spaces and $\mathcal{T}(H)$, $\mathcal{T}(K)$ be the trace classes on H, K respectively. Then, a linear map $\Delta:\mathcal{T}(H)\to\mathcal{T}(K)$ is completely bounded if and only if there exist operator sequences $\{A_i\}_i\subset\mathcal{B}(H,K)$ and $\{B_i\}_i\subset\mathcal{B}(K,H)$ satisfying $\|\sum_i A_i^\dagger A_i\|<\infty$, and $\|\sum_i B_iB_i^\dagger\|<\infty$ such that*

$$\Delta(T)=\sum_i A_iTB_i$$

for all $T\in\mathcal{T}(H)$.

Proof. If Δ has the form stated in the theorem, it is obvious that, for any $X\in\mathcal{B}(K)$,

$$\begin{aligned}\mathrm{Tr}(\textstyle\sum_i A_iTB_iX) =& \textstyle\sum_i \mathrm{Tr}(A_iTB_iX)\\ =& \textstyle\sum_i \mathrm{Tr}(TB_iXA_i)=\mathrm{Tr}(\sum_i TB_iXA_i)\end{aligned}$$

holds for all $T\in\mathcal{T}(H)$, so $\Delta^*(X)=\sum_i B_iXA_i\in\mathcal{B}(H)$. As $\|\sum_i A_i^\dagger A_i\|<\infty$, and $\|\sum_i B_iB_i^\dagger\|<\infty$, Δ^* is completely bounded with $\|\Delta^*\|_{\rm cb}\leq\|(\sum_i A_i^\dagger A_i)^{\frac12}\|\cdot\|(\sum_i B_iB_i^\dagger)^{\frac12}\|$. But $\|\Delta_n\|=\|\Delta_n^*\|$ (ref. [19, Proposition 3.2.2]). So, Δ is completely bounded.

Conversely, assume that $\Delta:\mathcal{T}(H)\to\mathcal{T}(K)$ is a completely bounded linear map; then $\Delta^*:\mathcal{B}(K)\to\mathcal{B}(H)$ is a completely bounded normal linear map. By Lemma 2.1, Δ^* is a generalized elementary operator. So there exists operator sequences $\{A_i\}_i \subset \mathcal{B}(H,K)$ and $\{B_i\}_i \subset \mathcal{B}(K,H)$ satisfying $\| \sum_i A_i^\dagger A_i \| < \infty$, and $\| \sum_i B_i B_i^\dagger \| < \infty$ such that $\Delta^*(X) = \sum_i B_i X A_i$ holds for all $X \in \mathcal{B}(K,H)$. Now, it is clear that $\Delta(T) = \sum_i A_i T B_i$ holds for all $T \in \mathcal{B}(K,H)$, completing the proof. □

By Lemma 3.1 and Theorem 2.5 the following results are immediate.

Theorem 3.2. *Let H, K be separable complex Hilbert spaces and $\mathcal{T}(H)$, $\mathcal{T}(K)$ be the trace classes on H, K respectively. Let $\Delta : \mathcal{T}(H) \to \mathcal{T}(K)$ be a linear map. Then*

(i) *Δ is positive and completely bounded if and only if there exist operator sequences $\{A_i\}_i \subset \mathcal{B}(H,K)$ and $\{B_i\}_i \subset \mathcal{B}(H,K)$ with $\| \sum_i A_i^\dagger A_i \| < \infty$ and $\| \sum_i B_i^\dagger B_i \| < \infty$, and a map $\Omega : H \to \mathcal{B}_1(l_2)$ such that $\mathbf{B}^\dagger|\psi\rangle = \Omega(|\psi\rangle)\mathbf{A}^\dagger|\psi\rangle$ for every $|\psi\rangle \in H$ and*

$$\Delta(T) = \sum_i A_i T A_i^\dagger - \sum_i B_j T B_j^\dagger$$

for all $T \in \mathcal{T}(H)$.

(ii) *Δ is completely positive if and only if there exist operator sequences $\{A_i\}_i \subset \mathcal{B}(H,K)$ with $\| \sum_i A_i^\dagger A_i \| < \infty$ such that*

$$\Delta(T) = \sum_i A_i T A_i^\dagger$$

for all $T \in \mathcal{T}(H)$.

Mathematically, like that for finite dimensional case, we may define a quantum channel (operation) as a trace-preserving (trace-decreasing) completely positive linear map from a trace-class into a trace-class. Thus by Theorem 3.2, we have

Corollary 3.3. *Every quantum channel (operation) $\mathcal{E}$ between two infinite-dimensional systems respectively associated with Hilbert spaces H and K has the form*

$$\mathcal{E}(\rho) = \sum_{i=1}^{\infty} M_i \rho M_i^\dagger,$$

where $\{M_i\} \subset \mathcal{B}(H,K)$ satisfies that $\sum_{i=1}^{\infty} M_i^\dagger M_i = I_H$ ($\sum_{i=1}^{\infty} M_i^\dagger M_i \leq I_H$).

Remark 3.4. For infinite dimensional case, is physically every trace-preserving completely positive linear map qualified being a quantum channel? Let $\mathcal{E} : \mathcal{T}(H) \to \mathcal{T}(H)$ be a trace-preserving positive linear map. If H is finite dimensional, and if $\mathcal{E}(\rho) = \sum_{i=1}^{k} E_i \rho E_i^\dagger$ for every ρ, then there exists a finite dimensional Hilbert space H_{env} with $\dim H_{\text{env}} = k$, and a unitary operator $U : H \otimes H_{\text{env}} \to H \otimes H_{\text{env}}$ such that Eq.(3.1) holds, that is, $\mathcal{E}(\rho) = \text{Tr}_{\text{env}}[U(\rho \otimes \rho_{\text{env}})U^\dagger]$ for all $\rho \in \mathcal{S}(H)$. In fact the unitary operator $U = (U_{ij})$ on $H \otimes H_{\text{env}}$ can be chosen so that $E_i = U_{i1}$ for each $i = 1, 2, \ldots, k$. This means that every trace-preserving completely positive linear map is a quantum channel. It is clear that this is not always true for the infinite dimensional case. In fact, let H be a Hilbert space with $\dim H = \infty$ and $\mathcal{E} : \mathcal{T}(H) \to \mathcal{T}(H)$ be a trace-preserving completely positive linear map defined by $\mathcal{E}(\rho) = \sum_{i=1}^{k} M_i \rho M_i^\dagger$ with $\sum_{i=1}^{k} M_i^\dagger M_i = I$, where $M_i \neq 0$ for each i and $k \leq \infty$; then there exist H_{env} with $\dim H_{\text{env}} = k$, $\rho_{\text{env}} \in \mathcal{S}(H_{\text{env}})$ and a unitary operator $U : H \otimes H_{\text{env}} \to H \otimes H_{\text{env}}$

such that Eq.(3.1) holds if and only if $\dim\ker\mathbf{M} = \dim\ker\mathbf{M}^\dagger$, where

$$\mathbf{M} = \begin{pmatrix} M_1 & 0 & 0 & \cdots \\ M_2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ M_i & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is an operator from H^k into H^k. Obviously, $\dim\ker\mathbf{M} = \infty \geq \dim\ker\mathbf{M}^\dagger$, and $\ker\mathbf{M}^\dagger = \{|\mathbf{y}\rangle = (\ |y_1\rangle \quad |y_2\rangle \quad \cdots \quad |y_i\rangle \quad \cdots\ )^T : \sum_i M_i^\dagger |y_i\rangle = 0\}$. Decompose the space H into $H = \bigoplus_{i=1}^k H_i$ so that $\dim H_i = \infty$ for each i and take $M_i : H \to H$ so that $M_i^\dagger$ is an isometry with range H_i. Then $\sum_i M^\dagger M_i = I$. As $\ker\mathbf{M}^\dagger = \{0\}$ we see that $\dim\ker\mathbf{M} \neq \dim\ker\mathbf{M}^\dagger$. However, if we allow that $\mathcal{E}(\rho) = \sum_{i=1}^k M_i\rho M_i^\dagger + M_\infty \rho M_\infty^\dagger$ with $M_\infty = 0$, and $\dim H_{\rm env} = k+1$, then there exists a unitary operator U such that U has the form

$$U = \begin{pmatrix} M_1 & U_{12} & \cdots & U_{1j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ M_i & U_{i2} & \cdots & U_{ij} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ M_\infty & U_{k+1,2} & \cdots & U_{k+1,j} & \cdots \end{pmatrix}$$

So, for some suitable state $\rho_{\rm env}$ the Eq.(3.1) holds. Thus, like the finite dimensional case, we still have that every trace-preserving completely positive linear map is a quantum channel.

4. Elementary operator criterion of separability of quantum states

Using the characterization of positive maps that are NCP in Section 2, we can establish a necessary and sufficient criterion of separability of states, that is, the elementary operator criterion.

The following necessary and sufficient condition for a state on finite dimensional spaces to be entangled is an immediate consequence of Corollary 2.7 and Horodeckis' Theorem.

Theorem 4.1. *Let H and K be finite dimensional complex Hilbert spaces and ρ be a state acting on $H \otimes K$. Then ρ is an entangled state if and only if there exists a linear map of the form $\Phi(\cdot) = \sum_{i=1}^k C_i(\cdot)C_i^\dagger - \sum_{j=1}^l D_j(\cdot)D_j^\dagger : \mathcal{B}(H) \to \mathcal{B}(K)$ with $\{D_1, \ldots, D_l\}$ a contractive locally linear combination of $\{C_1, \ldots, C_k\}$, such that the operator $(\Phi \otimes I)\rho$ is not positive.*

We will show below that this result is also true for infinite dimensional case. Before doing this, we write directly from Theorem 2.5 and Corollary 2.6 two sufficient criteria of entanglement of states for infinite dimensional systems.

Proposition 4.2. *Let H, K be complex Hilbert spaces and ρ be a state on $H \otimes K$. Then ρ is entangled if there exists an elementary operator of the form $\Phi(\cdot) = \sum_{i=1}^k C_i(\cdot)C_i^\dagger - \sum_{j=1}^l D_j(\cdot)D_j^\dagger : \mathcal{B}(H) \to \mathcal{B}(K)$, where $\{D_1, \ldots, D_l\}$ is a contractive locally linear combination but not a contractive linear combination of $\{C_1, \ldots, C_k\}$, such that the operator $(\Phi \otimes I)\rho$ is not positive.*

More generally, we have

Proposition 4.3. *Let H, K be complex Hilbert spaces and ρ be a state on $H \otimes K$. Then ρ is an entangled state if there exists a generalized elementary operator Φ defined by*

$$\Phi(X) = \sum_i A_i X A_i^\dagger - \sum_j C_j X C_j^\dagger$$

for every $X \in \mathcal{B}(H)$, where $\| \sum_i A_i A_i^\dagger \| < \infty$ and $\| \sum_j C_j C_j^\dagger \| < \infty$, $\{C_j\}_j$ is a generalized contractive locally linear combination but not a generalized contractive linear combination of $\{A_i\}_i$, such that $(\Phi \otimes I)\rho$ is not positive.

Propositions 4.2 and 4.3 only provide sufficient conditions for a state to be entangled. In fact, these conditions are also necessary, and thus we obtain a necessary and sufficient criterion for entanglement which we will call the elementary operator criterion. Much better can be reached. Note that an elementary operator Φ is of finite rank if and only if there exist finite rank operators A_i, B_i, $i = 1, 2, \cdots, k$, such that $\Phi(X) = \sum_{i=1}^k A_i X B_i$ [27]. We will prove that every entangled state can be detected by a positive elementary operator of finite rank.

Theorem 4.4. (Elementary operator criterion) *Let H, K be complex Hilbert spaces and ρ be a state on $H \otimes K$. Then ρ is entangled if and only if there exists an elementary operator of the form $\Phi(\cdot) = \sum_{i=1}^k C_i(\cdot)C_i^\dagger - \sum_{j=1}^l D_j(\cdot)D_j^\dagger : \mathcal{B}(H) \to \mathcal{B}(K)$, where all C_is and D_js are of finite rank and $\{D_1, \ldots, D_l\}$ is a contractive locally linear combination of $\{C_1, \ldots, C_k\}$, such that the operator $(\Phi \otimes I)\rho$ is not positive.*

Proof. The "if" part follows from Proposition 4.2. For the "only if" part, assume that the state ρ is inseparable. Take any orthonormal bases $\{|i\rangle\}$ and $\{|j\rangle\}$ of H and K, respectively. For any positive integers $s \leq \dim H$ and $t \leq \dim K$, denote $P_{st} = P_s \otimes Q_t$, where P_s and Q_t are finite rank projections onto the subspaces H_s and K_t spanned by $\{|i\rangle\}_{i=0}^s$ and $\{|j\rangle\}_{j=0}^t$, respectively. Since ρ is entangled, by [42, Theorem 2], there exists (s, t) such that $\rho_{st} = \mathrm{Tr}(P_{st}\rho P_{st})^{-1} P_{st}\rho P_{st}$ is entangled. Regarding ρ_{st} as a state on $H_s \otimes K_t$. As $\dim(H_s \otimes K_t) < \infty$, by Theorem 4.1, there exists a positive map $\Delta : \mathcal{B}(H_s) \to \mathcal{B}(K_t)$ of the form $\Delta(\cdot) = \sum_{i=1}^k A_i(\cdot)A_i^\dagger - \sum_{j=1}^l B_j(\cdot)B_j^\dagger$ with $\{B_1, \ldots, B_l\}$ a contractive locally linear combination but not a contractive linear combination of $\{A_1, \ldots, A_k\}$, such that the operator $(\Delta \otimes Q_t)\rho_{st}$ is not positive on $K_t \otimes K_t$. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be defined by $\Phi(X) = Q_t \Delta(P_s X P_s) Q_t$. Then Φ is positive and $\Phi(X) = \sum_{i=1}^k C_i(X)C_i^\dagger - \sum_{j=1}^l D_j(X)D_j^\dagger$, where $C_i = Q_t A_i P_s$ and $D_j = Q_t B_i P_s$ are of finite rank.

Represent ρ as an operator matrix $\rho = (\eta_{ij})_{i,j}$ according to the bases $\{|i\rangle\}_{i=0}^s$ and $\{|j\rangle\}_{j=0}^t$, where $\eta_{ij} \in \mathcal{B}(H)$. Obviously,

$$\rho_{st} = \mathrm{Tr}(P_{st}\rho P_{st})^{-1} \begin{pmatrix} P_s\eta_{11}P_s & P_s\eta_{12}P_s & \cdots & P_s\eta_{1t}P_s \\ P_s\eta_{21}P_s & P_s\eta_{22}P_s & \cdots & P_s\eta_{2t}P_s \\ \vdots & \vdots & \ddots & \vdots \\ P_s\eta_{t1}P_s & P_s\eta_{t2}P_s & \cdots & P_s\eta_{tt}P_s \end{pmatrix}.$$

Thus we have

$$(\Delta\otimes Q_t)\rho_{st}=\mathrm{Tr}(P_{st}\rho P_{st})^{-1}\begin{pmatrix}\Delta(P_s\eta_{11}P_s) & \Delta(P_s\eta_{12}P_s) & \cdots & \Delta(P_s\eta_{1t}P_s)\\ \Delta(P_s\eta_{21}P_s) & \Delta(P_s\eta_{22}P_s) & \cdots & \Delta(P_s\eta_{2t}P_s)\\ \vdots & \vdots & \ddots & \vdots\\ \Delta(P_s\eta_{t1}P_s) & \Delta(P_s\eta_{t2}P_s) & \cdots & \Delta(P_s\eta_{tt}P_s)\end{pmatrix} \tag{4.1}$$

is not positive. Note that $\Phi(\eta_{ij})=Q_t\Delta(P_s\eta_{ij}P_s)Q_t=\Delta(P_s\eta_{ij}P_s)$. So

$$(\Phi\otimes I)\rho$$
$$=\left(\begin{array}{cccc|cc}\Delta(P_s\eta_{11}P_s) & \Delta(P_s\eta_{12}P_s) & \cdots & \Delta(P_s\eta_{1t}P_s) & \Delta(P_s\eta_{1(t+1)}P_s) & \cdots\\ \Delta(P_s\eta_{21}P_s) & \Delta(P_s\eta_{22}P_s) & \cdots & \Delta(P_s\eta_{2t}P_s) & \Delta(P_s\eta_{2(t+1)}P_s) & \cdots\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots\\ \Delta(P_s\eta_{t1}P_s) & \Delta(P_s\eta_{t2}P_s) & \cdots & \Delta(P_s\eta_{tt}P_s) & \Delta(P_s\eta_{t(t+1)}P_s) & \cdots\\ \hline \Delta(P_s\eta_{(t+1)1}P_s) & \Delta(P_s\eta_{(t+1)2}P_s) & \cdots & \Delta(P_s\eta_{(t+1)t}P_s) & \Delta(P_s\eta_{(t+1)(t+1)}P_s) & \cdots\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots\end{array}\right).$$

It follows that $(\Phi\otimes I)\rho$ is not positive since it has a non positive $t\times t$ submatrix (4.1). The proof is completed. □

To sum up, we have proved the following criterion of separability, which is valid for both finite and infinite dimensional systems, improves Stømer's theorem [44] and is easier to be handled by our characterization of positive elementary operators.

Theorem 4.5. (Elementary operator criterion) *Let H, K be complex Hilbert spaces and ρ be a state acting on $H\otimes K$. Then the following statements are equivalent.*

(1) *ρ is separable;*

(2) *$(\Phi\otimes I)\rho\geq 0$ holds for every positive elementary operator $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$.*

(3) *$(\Phi\otimes I)\rho\geq 0$ holds for every finite-rank positive elementary operator $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$.*

5. Some examples of constructing NCP positive maps

It follows from Theorem 4.4, 4.5 and Theorem 2.11, for both finite and infinite dimensional systems, it is very important to construct NCP positive linear maps between matrix algebras since the non-complete positivity of a positive elementary operator is essentially determined by its behavior on finite-dimensional subspaces. In this section we give some concrete examples of NCP positive linear maps between matrix algebras by applying the results in Section 2.

Let H be a complex Hilbert space of $\dim H=n<\infty$ and let $\{|1\rangle,|2\rangle,\ldots,|n\rangle\}$ be an orthonormal basis of H. Denote $E_{ij}=|i\rangle\langle j|$, $1\leq i,j\leq n$. The well known NCP positive map on $\mathcal{B}(H)$, that is, the transpose $T\mapsto T^t$ is an elementary operator

$$T^t=\sum_{i=1}^n E_{ii}TE_{ii}+\sum_{i<j}A_{ij}TA_{ij}^\dagger-\sum_{i<j}C_{ij}TC_{ij}^\dagger\quad\forall T,$$

where $A_{ij}=\frac{1}{\sqrt{2}}(E_{ij}+E_{ji})$, $C_{ij}=\frac{1}{\sqrt{2}}(E_{ij}-E_{ji})$. Another example of well known NCP positive map is the reduction map, which has the form

$$T\mapsto \mathrm{Tr}(T)I-T=\sum_{i\neq j}E_{ij}TE_{ji}+\sum_{i\neq j}G_{ij}AG_{ij}^{\dagger}-\sum_{i\neq j}F_{ij}AF_{ij}^{\dagger}\quad \forall T,$$

where $F_{ij}=\frac{1}{\sqrt{2}}(E_{ii}+E_{jj})$ and $G_{ij}=\frac{1}{\sqrt{2}}(E_{ii}-E_{jj})$.

Next we give another kind of NCP positive linear maps.

Proposition 5.1. *Let H be a complex Hilbert space of $2\leq \dim H=n<\infty$ and let $\{|1\rangle,|2\rangle,\ldots,|n\rangle\}$ be an orthonormal basis of H. Denote $E_{ij}=|i\rangle\langle j|$, $1\leq i,j\leq n$. Let $A_k=\sum_{i=1}^{n}a_{ki}|i\rangle\langle i|$, $k=1,\ldots,s$ and $B_l=\sum_{i=1}b_{li}|i\rangle\langle i|$, $l=1,\ldots,t$ with $t>0$ and $s+t\leq n$. Assume that $\{A_k,B_l:k=1,\ldots,s;l=1,\ldots,t\}$ is a linearly independent set. Let $\Delta:\mathcal{B}(H)\to\mathcal{B}(H)$ be the linear map defined by*

$$\Delta(T)=\sum_{k=1}^{s}A_kTA_k^{\dagger}+\sum_{i\neq j}E_{ij}TE_{ij}^{\dagger}-\sum_{l=1}^{t}B_lTB_l^{\dagger} \tag{5.1}$$

for every $T\in\mathcal{B}(H)$. If $\sum_{k=1}^{s}|a_{ki}|^2\geq\sum_{l=1}^{t}|b_{li}|^2$, $|\sum_{k=1}^{s}a_{ki}a_{kj}-\sum_{l=1}^{s}b_{li}b_{lj}|\leq 1$ whenever $i\neq j$, then Δ is NCP positive.

Proof. It is clear that Δ defined in Eq.(5.1) is not completely positive since B_j is linearly independent to $\{A_k,E_{ij}:1\leq k\leq s;1\leq i,j\leq n,i\neq j\}$. Assume that $\sum_{k=1}^{s}|a_{ki}|^2\geq\sum_{l=1}^{t}|b_{li}|^2$, $|\sum_{k=1}^{s}a_{ki}a_{kj}-\sum_{l=1}^{s}b_{li}b_{lj}|\leq 1$ whenever $i\neq j$, We will show that Δ is positive.

Note that

$$\Delta(E_{mm})=(\sum_{k=1}^{s}|a_{km}|^2-\sum_{l=1}^{t}|b_{lm}|^2)E_{kk}+\sum_{i\neq k}E_{ii} \tag{5.2}$$

and

$$\Delta(E_{ij})=(\sum_{k=1}^{s}a_{ki}\bar{a}_{kj}-\sum_{l=1}^{t}b_{li}\bar{b}_{lj})E_{ij}\quad \text{if } i\neq j. \tag{5.3}$$

Let $f_{ii}=\sum_{k=1}^{s}|a_{ki}|^2-\sum_{l=1}^{t}|b_{li}|^2$ and $f_{ij}=\sum_{k=1}^{s}a_{ki}\bar{a}_{kj}-\sum_{l=1}^{t}b_{li}\bar{b}_{lj}$ if $i\neq j$. Clearly, $f_{ji}=\bar{f}_{ij}$ for all i,j.

Identify H with $\mathbb{C}^n$. For any $|\psi\rangle = (\xi_1, \xi_2, \ldots, \xi_n)^T \in \mathbb{C}^n$, consider the rank-one positive matrix $|\psi\rangle\langle\psi| = (\xi_i\bar{\xi_j})$. By Eqs.(5.2) and (5.3) we have

$$\begin{array}{rcl}\Delta(|\psi\rangle\langle\psi|) &=& \begin{pmatrix} f_{11}|\xi_1|^2 & f_{12}\xi_1\bar{\xi_2} & \cdots & f_{1n}\xi_1\bar{\xi_n} \\ f_{21}\xi_2\bar{\xi_1} & f_{22}|\xi_2|^2 & \cdots & f_{2n}\xi_2\bar{\xi_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}\xi_n\bar{\xi_1} & f_{n2}\xi_n\bar{\xi_2} & \cdots & f_{nn}|\xi_n|^2 \end{pmatrix} \\ && + \begin{pmatrix} \sum_{1\leq j\leq n, j\neq 1}|\xi_j|^2 & 0 & \cdots & 0 \\ 0 & \sum_{1\leq j\leq n, j\neq 2}|\xi_j|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{1\leq j\leq n, j\neq n}|\xi_j|^2 \end{pmatrix} \\ &\geq& \begin{pmatrix} \sum_{1\leq j\leq n, j\neq 1}|\xi_j|^2 & f_{12}\xi_1\bar{\xi_2} & \cdots & f_{1n}\xi_1\bar{\xi_n} \\ f_{21}\xi_2\bar{\xi_1} & \sum_{1\leq j\leq n, j\neq 2}|\xi_j|^2 & \cdots & f_{2n}\xi_2\bar{\xi_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}\xi_n\bar{\xi_1} & f_{n2}\xi_n\bar{\xi_2} & \cdots & \sum_{1\leq j\leq n, j\neq n}|\xi_j|^2 \end{pmatrix} \\ &=& C_\psi \end{array}$$

So it suffices to show that $C_\psi \geq 0$.

To do this, denote $c_i = |\xi_i|$. Then, by the assumption of $|f_{ij}| \leq 1$ for $i \neq j$, we have $f_{ij}\xi_i\bar{\xi_j} = c_ic_jv_{ij}$ with $|v_{ij}| \leq 1$, and

$$C_\psi = \begin{pmatrix} \sum_{1\leq j\leq n, j\neq 1} c_j^2 & c_1c_2v_{12} & \cdots & c_1c_nv_{1n} \\ c_1c_2\bar{v_{12}} & \sum_{1\leq j\leq n, j\neq 2} c_j^2 & \cdots & c_2c_nv_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_1c_n\bar{v_{1n}} & c_2c_n\bar{v_{2n}} & \cdots & \sum_{1\leq j\leq n, j\neq n} c_j^2 \end{pmatrix}.$$

For any $|\phi\rangle = (\eta_1, \eta_2, \ldots, \eta_n)^T \in \mathbb{C}^n$, writing $d_i = |\eta_i|$, we have

$$\begin{array}{rcl}\langle\phi|C_\psi|\phi\rangle &=& \sum_{i=1}^n(\sum_{1\leq j\leq n, j\neq i} c_j^2)|\eta_i|^2 + 2\mathrm{Re}(\sum_{i<j} c_ic_jv_{ij}\eta_j\bar{\eta_i}) \\ &\geq& \sum_{i=1}^n(\sum_{1\leq j\leq n, j\neq i} c_j^2)d_i^2 - 2\sum_{i<j} c_ic_jd_id_j \\ &=& \sum_{i<j}(c_id_j - c_jd_i)^2 \geq 0. \end{array}$$

Therefore, $C_\psi \geq 0$. We have proved that $\Delta(|\psi\rangle\langle\psi|) \geq 0$ holds for all rank-one positive matrices $|\psi\rangle\langle\psi|$. It follows that Δ is a positive linear map, as desired. □

In the following, we give some preliminary results on characterizing positive elementary operators, which are needed in later.

The following result is easily checked and is useful to us.

Proposition 5.2. *Let*

$$B_{(t_1,t_2\cdots,t_n)} = \begin{pmatrix} t_1 & -1 & -1 & \cdots & -1 \\ -1 & t_2 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & t_n \end{pmatrix} \in M_n(\mathbb{C}).$$

If $t_i \geq n-1$ for each $i = 1, 2, \cdots, n$, then $B_{(t_1,t_2\cdots,t_n)} \geq 0$ (that is, $B_{(t_1,t_2\cdots,t_n)}$ is semi-positive definite); if $t_i < n-1$ for each $i = 1, 2, \cdots, n$, then $B_{(t_1,t_2\cdots,t_n)} \ngeq 0$. Particularly, $B_{(t,t,\cdots,t)} \geq 0$ if and only if $t \geq n-1$.

Proof. Assume that $t_i \geq n-1$ for each $i = 1, 2, \cdots, n$. Then $t_0 = \min\{t_1, t_2, \cdots, t_n\} \geq n-1$. For any $|x\rangle = (\xi_1, \xi_2, \ldots, \xi_n)^T \in \mathbb{C}^n$, we have

$$\begin{aligned}
\langle x|B_{(t_1,t_2\cdots,t_n)}|x\rangle = & \ t_0 \sum_{i=1}^n |\xi_i|^2 - 2\sum_{i<j} \xi_i \bar{\xi_j} \\
\geq & \ t_0 \sum_{i=1}^n |\xi_i|^2 - 2\sum_{i<j} |\xi_i||\xi_j| \\
= & \ (t_0 - n + 1)\sum_{i=1}^n |\xi_i|^2 + (n-1)\sum_{i=1}^n |\xi_i|^2 - 2\sum_{i<j} |\xi_i||\xi_j| \\
= & \ (t_0 - n + 1)\sum_{i=1}^n |\xi_i|^2 + \sum_{i<j}^n (|\xi_i| - |\xi_j|)^2 \geq 0,
\end{aligned}$$

which implies that $B_{(t_1,t_2\cdots,t_n)} \geq 0$. If $t_i < n-1$ for each $i = 1, 2, \cdots, n$, then $t_0' = \max\{t_1, t_2, \cdots, t_n\} < n-1$. Taking $\xi_1 = \xi_2 = \cdots = \xi_n \neq 0$ and let $|x_0\rangle = (\xi_1, \xi_1, \ldots, \xi_1)^T$, one gets $\langle x_0|B_{(t_1,t_2\cdots,t_n)}|x_0\rangle \leq (t_0' - n + 1)n\sum_{i=1}^n |\xi_1|^2 < 0$. It follows that $B_t \ngeq 0$, completing the proof. □

There is another simple proof of Proposition 5.2 suggested by Chi-Kwong Li by applying the fact that an operator $A = D - |\psi\rangle\langle\psi|$ with $D \geq 0$ invertible is positive if and only if $\|D^{-\frac{1}{2}}|\psi\rangle\| \leq 1$.

By using of above results, we can prove the following result.

Proposition 5.3. *Let H and K be Hilbert spaces and let $\{|i\rangle\}_{i=1}^n$ and $\{|i'\rangle\}_{i=1}^n$ be any orthonormal sets of H and K, respectively. Denote $E_{ji} = |j'\rangle\langle i| \in \mathcal{B}(H, K)$. Let $\Delta : \mathcal{B}(H) \to \mathcal{B}(K)$ be defined by*

$$\Delta_{(t_1,t_2,\cdots,t_n)}(A) = \ \sum_{i=1}^n t_i E_{ii} A E_{ii}^\dagger - (\sum_{i=1}^n E_{ii}) A (\sum_{i=1}^n E_{ii})^\dagger$$

for all $A \in \mathcal{B}(H)$. If $t_i \geq n$ for each $i = 1, 2, \cdots, n$, then $\Delta_{(t_1,t_2,\cdots,t_n)}$ is a completely positive map; if $t_i < n$ for each $i = 1, 2, \cdots, n$, then $\Delta_{(t_1,t_2,\cdots,t_n)}$ is not a positive map. Particularly, $\Delta_{(t,t,\cdots,t)}$ is positive if and only if it is completely positive, and in turn, if and only if $t \geq n$.

Proof. For any unit vector $|x\rangle = (\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots)^T \in H$, consider the rank-one projection $|x\rangle\langle x|$. We have

$$\Delta(|x\rangle\langle x|) = \left(\begin{array}{cccc|ccc}
(t_1-1)|\xi_1|^2 & -\xi_1\bar{\xi_2} & \cdots & -\xi_1\bar{\xi_n} & 0 & 0 & \cdots \\
-\xi_2\bar{\xi_1} & (t_2-1)|\xi_2|^2 & \cdots & -\xi_2\bar{\xi_n} & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-\xi_n\bar{\xi_1} & -\xi_n\bar{\xi_2} & \cdots & (t_n-1)|\xi_n|^2 & 0 & 0 & \cdots \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{array}\right). \quad (5.4)$$

If $t_i < n$ for each $i = 1, 2, \cdots, n$, taking $|x\rangle = (1, 1, \ldots, 1, 0, 0, \cdots)^T$ in Eq.(5.4) and by Proposition 5.2, we get $\Delta(|x\rangle\langle x|) \ngeq 0$, and so Δ is not positive.

On the other hand, assume that $t_i \geq n$ for each $i = 1, 2, \cdots, n$. Since $\sum_{i=1}^n E_{ii} = \sum_{i=1}^n \frac{1}{\sqrt{t_i}}(\sqrt{t_i}E_{ii})$ and $\sum_{i=1}^n (\frac{1}{\sqrt{t_i}})^2 \leq \sum_{i=1}^n (\frac{1}{\sqrt{n}})^2 \leq 1$, $\sum_{i=1}^n E_{ii}$ is a contractive linear combination of $\{\sqrt{t_1}E_{11}, \sqrt{t_2}E_{22}, \cdots, \sqrt{t_n}E_{nn}\}$. By Corollary 2.6, Δ is completely positive. □

For the sake of convenience, we introduce a terminology here.

Definition 5.4. Let $\Delta : \mathcal{B}(H) \to \mathcal{B}(K)$ be a finite rank elementary operator. It follows from a characterization of finite rank elementary operators in [27] that there exist finite rank projections $P \in \mathcal{B}(H)$ and $Q \in \mathcal{B}(K)$ such that

$$\Delta(A) = Q\Delta(PAP)Q \text{ for all } A \in \mathcal{B}(H). \tag{5.5}$$

Let

$$(n, m) = \min\{(\text{rank}(P), \text{rank}(Q)) : (P, Q) \text{ satisfies the equation (5.5)}\}.$$

(n, m) is called the order of Δ, and we say that the elementary operator Δ is of the order (n, m).

6. Positive finite rank elementary operators of order (2,2) and (3,3)

In this section we will construct some positive finite rank elementary operators of order $(2, 2)$ and $(3, 3)$. Applying such positive maps, we give a simple necessary and sufficient condition for a pure state to be separable. We also use these positive maps to detect some entangled mixed states.

Positive elementary operators of order $(2, 2)$ are easily constructed. For example, Let H and K be Hilbert spaces of dimension ≥ 2, and let $\{|i\rangle\}_{i=1}^2$ and $\{|j'\rangle\}_{j=1}^2$ be any orthonormal sets of H and K, respectively. Let $\Phi_0 : \mathcal{B}(H) \to \mathcal{B}(K)$ be defined by

$$\begin{aligned} \Phi_0(A) = \quad & E_{11}AE_{11}^\dagger + E_{22}AE_{22}^\dagger + E_{12}AE_{12}^\dagger \\ & + E_{21}AE_{21}^\dagger - (E_{11} + E_{22})A(E_{11} + E_{22})^\dagger \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \Psi_0(A) = \quad & (2E_{11} + E_{22})A(2E_{11} + E_{22})^\dagger + E_{12}AE_{12}^\dagger \\ & + E_{21}AE_{21}^\dagger - (E_{11} + E_{22})A(E_{11} + E_{22})^\dagger \end{aligned} \tag{6.2}$$

for every $A \in \mathcal{B}(H)$, where $E_{ji} = |j'\rangle\langle i|$. It is obvious that both Φ_0 and Ψ_0 are positive because the map

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

and the map

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} 3a_{11} + a_{22} & a_{12} \\ a_{21} & a_{11} \end{pmatrix}$$

on $M_2(\mathbb{C})$ are positive. A surprising fact is that such simple positive elementary operators of order $(2, 2)$ will be enough to determine the separability of the pure states.

Let $\mathcal{U}(H)$ (resp. $\mathcal{U}(K)$) be the group of all unitary operators on H (resp. on K). For any map $\Delta : \mathcal{B}(H) \to \mathcal{B}(K)$ and any unitary operators $U \in \mathcal{U}(H)$ and $V \in \mathcal{U}(K)$, the deduced map $A \mapsto V^\dagger\Delta(U^\dagger AU)V$ will be denoted by $\Delta^{U,V}$. Though there is no "universal" NCP positive linear map that can recognize all entangled states, we still hope that the following conjecture is true:

Conjecture 6.0. *There exists a NCP positive linear map Δ that is universal in the sense that the set $\mathcal{G}(\Delta)=\{\Delta^{U,V}: U\in\mathcal{U}(H), V\in\mathcal{U}(K)\}$ provides a necessary and sufficient criterion of separability.*

We do not know if this conjecture is true. But the next result is a support of the conjecture, which gives a simple necessary and sufficient criterion of separability for *pure states* in bipartite composite systems of any dimension, by $\mathcal{G}(\Phi)$ with Φ a suitable elementary operator or order (2,2).

Theorem 6.1. *Let H and K be Hilbert spaces of dimension ≥ 2, and let $\{|i\rangle\}_{i=1}^2$ and $\{|j'\rangle\}_{j=1}^2$ be any orthonormal sets of H and K, respectively. Let $\Phi_0(\Psi_0):\mathcal{B}(H)\to\mathcal{B}(K)$ be defined by Eq.(6.1) (Eq.(6.2)). Then a pure state ρ on $H\otimes K$ is separable if and only if*

$$(\Phi_0^{U,V}\otimes I)\rho\geq 0 \ ((\Psi_0^{U,V}\otimes I)\rho\geq 0)$$

holds for all $U\in\mathcal{U}(H)$ and $V\in\mathcal{U}(K)$.

Proof. If a state ρ is separable, then $(\Phi_0^{U,V}\otimes I)\rho\geq 0$ $((\Psi_0^{U,V}\otimes I)\rho\geq 0)$ as $\Phi_0^{U,V}$ $(\Psi_0^{U,V})$ is a positive map.

Conversely, assume that $\rho=|\psi\rangle\langle\psi|$ is an inseparable pure state. Let $|\psi\rangle=\sum_{k=1}^{N_\psi}\delta_k|k,k'\rangle$ be the Schmidt decomposition, where $\delta_1\geq\delta_2\geq\cdots>0$ with $\sum_{k=1}^{N_\psi}\delta_k^2=1$, and $\{|k\rangle\}_{k=1}^{N_\psi}$ and $\{|k'\rangle\}_{k=1}^{N_\psi}$ are orthonormal in H and K, respectively. Thus $\rho=\sum_{k,l=1}^{N_\psi}\delta_k\delta_{k'}|k,k'\rangle\langle l,l'|=\sum_{k,l=1}^{N_\psi}\delta_k\delta_{k'}E_{kl}\otimes E_{k'l'}$. Since $\rho=|\psi\rangle\langle\psi|$ is inseparable, the Schmidt number N_ψ of $|\psi\rangle$ is greater than 1 and hence $\delta_1\geq\delta_2>0$.

Up to unitary equivalence, we may assume that $\{|k\rangle\}_{k=1}^2=\{|i\rangle\}_{i=1}^2$ and $\{|k'\rangle\}_{k'=1}^2=\{|j'\rangle\}_{i=1}^2$. Then, since $\Phi_0(E_{kl})=0$ $(\Psi_0(E_{kl})=0)$ whenever $k>2$ or $l>2$, we have

$$\begin{aligned}(\Phi_0\otimes I)\rho= &\ \textstyle\sum_{i,j=1}^2\delta_i\delta_j\Phi_0(E_{ij})\otimes E_{ij}\\ \cong &\begin{pmatrix}0&0&0&-\delta_1\delta_2\\0&\delta_1^2&0&0\\0&0&\delta_2^2&0\\-\delta_1\delta_2&0&0&0\end{pmatrix}\oplus 0\end{aligned}$$

$$\begin{aligned}((\Psi_0\otimes I)\rho= &\ \textstyle\sum_{i,j=1}^2\delta_i\delta_j\Psi_0(E_{ij})\otimes E_{ij}\\ \cong &\begin{pmatrix}3\delta_1^2&0&0&\delta_1\delta_2\\0&\delta_1^2&0&0\\0&0&\delta_2^2&0\\\delta_1\delta_2&0&0&0\end{pmatrix}\oplus 0,\end{aligned}$$

which is clearly not positive. $\square$

Now let us consider the positive elementary operators of order $(3,3)$.

Theorem 6.2. *Let H and K be Hilbert spaces of dimension ≥ 3, and let $\{|i\rangle\}_{i=1}^3$ and $\{|j'\rangle\}_{j=1}^3$ be any orthonormal sets of H and K, respectively. Let $\Phi,\Phi':\mathcal{B}(H)\to\mathcal{B}(K)$ be* *defined by*

$$\begin{array}{rl}\Phi(A)= & 2\sum_{i=1}^{3}E_{ii}AE_{ii}^{\dagger}+E_{12}AE_{12}^{\dagger}+E_{23}AE_{23}^{\dagger}+E_{31}AE_{31}^{\dagger}\\ & -(\sum_{i=1}^{3}E_{ii})A(\sum_{i=1}^{3}E_{ii})^{\dagger}\end{array}\tag{6.3}$$

and

$$\begin{array}{rl}\Phi'(A)= & 2\sum_{i=1}^{3}E_{ii}AE_{ii}^{\dagger}+E_{13}AE_{13}^{\dagger}+E_{21}AE_{21}^{\dagger}+E_{32}AE_{32}^{\dagger}\\ & -(\sum_{i=1}^{3}E_{ii})A(\sum_{i=1}^{3}E_{ii})^{\dagger}\end{array}\tag{6.3$'$}$$

for every $A\in\mathcal{B}(H)$, *where* $E_{ji}=|j'\rangle\langle i|$. *Then* Φ *and* Φ' *are indecomposable positive finite rank elementary operators of order* $(3,3)$.

Proof. We only give the proof that Φ is NCP positive. The fact that Φ is not decomposable will be proved in Example 6.3. Φ' is dealt with similarly.

It is obvious that Φ is a finite rank elementary operator of order $(3,3)$. Also, it is clear from Corollary 2.6 that Φ is not completely positive because $\sum_{i=1}^{3}E_{ii}$ is not a contractive linear combination of

$$\{\sqrt{2}E_{11},\sqrt{2}E_{22},\sqrt{2}E_{33},E_{12},E_{23},E_{31}\}.$$

To prove the positivity of Φ, extend $\{|i\rangle\}_{i=1}^{3}$ and $\{|j'\rangle\}_{j=1}^{3}$ to orthonormal bases $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j'\rangle\}_{j=1}^{\dim K}$ of H and K, respectively. Then every $A\in\mathcal{B}(H)$ has a matrix representation $A=(a_{kl})$ and the map Φ maps A into

$$\Phi(A)=\begin{pmatrix} a_{11}+a_{22} & -a_{12} & -a_{13} & 0 & 0 & \cdots\\ -a_{21} & a_{22}+a_{33} & -a_{23} & 0 & 0 & \cdots\\ -a_{31} & -a_{32} & a_{33}+a_{11} & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix},$$

which is unitarily equivalent to

$$S\oplus 0=\begin{pmatrix} a_{11}+a_{22} & -a_{12} & -a_{13}\\ -a_{21} & a_{22}+a_{33} & -a_{23}\\ -a_{31} & -a_{32} & a_{33}+a_{11}\end{pmatrix}\oplus 0.$$

It is easily checked that (also see [29, Proposition 5.2]) the matrix S is positive. So $\Phi(A)$ is positive, completing the proof of the theorem. □

Next we use the positive maps in Theorem 6.2 to detect some mixed entangled states. The example also implies that the positive maps in Theorem 6.2 are not decomposable since they can recognize some PPT entangled states.

The states ρ_t in the next example were introduced in [30] firstly.

Example 6.3. Let H and K be complex Hilbert spaces of dimension ≥ 3 and let $\{|i\rangle\}_{i=1}^{3}$ and $\{|j'\rangle\}_{j=1}^{3}$ be any orthonormal sets of H and K, respectively. Let

$$|\omega_1\rangle=\frac{1}{\sqrt{3}}(|11'\rangle+|22'\rangle+|33'\rangle)\ \text{ and }\ |\omega_2\rangle=\frac{1}{\sqrt{3}}(|12'\rangle+|23'\rangle+|31'\rangle).$$

Define $\rho_1 = |\omega_1\rangle\langle\omega_1|$, $\rho_2 = |\omega_2\rangle\langle\omega_2|$ and $\rho_3 = \frac{1}{3}(|13'\rangle\langle 13'| + |21'\rangle\langle 21'| + |32'\rangle\langle 32'|)$. Let $\rho = \sum_{i=1}^{3} q_i\rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i = 1, 2, 3$ with $q_1 + q_2 + q_3 = 1$, $t \in [0, 1]$, and ρ_0 is a state on $H \otimes K$.

Hou and Qi in [30] proved that, if $q_2 < \frac{5}{7}q_1$ or $q_1 < \frac{5}{7}q_2$, then, for sufficiently small t, ρ_t is entangled; if $q_2 < \frac{5}{7}q_1$ or $q_1 < \frac{5}{7}q_2$, and if $q_1q_2q_3 \geq q_1^3 + q_2^3$, then ρ_t is PPT entangled whenever ρ_0 is. Now, by using of the positive finite rank elementary operators Φ and Φ' constructed in Theorem 6.2, we can give a finer result. In fact, for sufficiently small t, or for ρ_0 with $(\Phi \otimes I)\rho_0 = (\Phi' \otimes I)\rho_0 = 0$ (for example, taking $\rho_0 = \sum_{i=4}^{\infty} p_i|i\rangle\langle i'| \otimes |i\rangle\langle i'|,\ p_i \geq 0,\ \sum_{i=4}^{\infty} p_i = 1$), the following statements are true.

(1) If $q_1 \neq q_2$ or $q_1 = q_2 > q_3$, then ρ_t is entangled.

(2) Let ρ_0 be PPT. Then ρ_t is PPT if and only if $q_1q_2q_3 \geq q_1^3 + q_2^3$. Particularly, if $q_j = 2q_i$ and $\frac{9}{2}q_j \leq q_3$, where $i, j \in \{1, 2\}$ and $i \neq j$, then ρ_t is PPT entangled.

In fact, by [30], we need only to check the following:

$(1)'$ If $q_1 \neq q_2$ or $q_1 = q_2 > q_3$, then ρ is entangled.

$(2)'$ ρ is PPT if and only if $q_1q_2q_3 \geq q_1^3 + q_2^3$. Particularly, if $q_j = 2q_i$ and $\frac{9}{2}q_j \leq q_3$, where $i, j \in \{1, 2\}$ and $i \neq j$, then ρ is PPT entangled.

For $\rho = q_1\rho_1 + q_2\rho_2 + q_3\rho_3$, it is obvious that

$$\rho = \frac{1}{3}\begin{pmatrix} q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1 \\ 0 & q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_2 & q_2 & 0 & 0 & 0 & q_2 & 0 \\ 0 & 0 & q_2 & q_2 & 0 & 0 & 0 & q_2 & 0 \\ q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 0 & q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_3 & 0 & 0 \\ 0 & 0 & q_2 & q_2 & 0 & 0 & 0 & q_2 & 0 \\ q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1 \end{pmatrix} \oplus 0.$$

Note that

$$\begin{aligned} & 3(\Phi \otimes I)(\rho) \\ \cong & \begin{pmatrix} q_1+q_3 & 0 & 0 & 0 & -q_1 & 0 & 0 & 0 & -q_1 \\ 0 & q_2+q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_1+q_2 & -q_2 & 0 & 0 & 0 & -q_2 & 0 \\ 0 & 0 & -q_2 & q_1+q_2 & 0 & 0 & 0 & -q_2 & 0 \\ -q_1 & 0 & 0 & 0 & q_1+q_3 & 0 & 0 & 0 & -q_1 \\ 0 & 0 & 0 & 0 & 0 & q_2+q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_2+q_3 & 0 & 0 \\ 0 & 0 & -q_2 & -q_2 & 0 & 0 & 0 & q_1+q_2 & 0 \\ -q_1 & 0 & 0 & 0 & -q_1 & 0 & 0 & 0 & q_1+q_3 \end{pmatrix} \oplus 0, \end{aligned}$$

which is unitarily equivalent to the operator $A \oplus B \oplus C \oplus 0$, where

$$A = \begin{pmatrix} q_1+q_3 & -q_1 & -q_1 \\ -q_1 & q_1+q_3 & -q_1 \\ -q_1 & -q_1 & q_1+q_3 \end{pmatrix}, \quad B = \begin{pmatrix} q_1+q_2 & -q_2 & -q_2 \\ -q_2 & q_1+q_2 & -q_2 \\ -q_2 & -q_2 & q_1+q_2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} q_2+q_3 & 0 & 0 \\ 0 & q_2+q_3 & 0 \\ 0 & 0 & q_2+q_3 \end{pmatrix} \geq 0.$$

For the matrices A and B, by Proposition 5.2, we get that $A \not\geq 0$ if $q_3 < q_1$ and $B \not\geq 0$ if $q_1 < q_2$. So $(\Phi \otimes I)(\rho)$ is not positive if $q_3 < q_1$ or $q_1 < q_2$. It follows from the elementary operator criterion Theorem 4.4 that ρ is entangled if $q_3 < q_1$ or $q_1 < q_2$. Note that ρ is PPT if and only if $q_1q_2q_3 \geq q_1^3 + q_2^3$. Thus, particularly, we obtain that ρ is PPT entangled if $q_2 = 2q_1$ and $\frac{9}{2}q_1 \leq q_3$.

Similarly, by applying the map Φ', one can get that the other half of the assertions $(1)'$-$(2)'$ is true.

7. Positive finite rank elementary operators of order $(4, 4)$

In this section we will construct some positive finite rank elementary operators of order $(4, 4)$. The following is our main result.

Theorem 7.1. *Let H and K be Hilbert spaces of dimension greater than 3 and let $\{|i\rangle\}_{i=1}^4$ and $\{|j'\rangle\}_{j=1}^4$ be any orthonormal sets of H and K, respectively. Let $\Phi, \Phi', \Phi'' : \mathcal{B}(H) \to \mathcal{B}(K)$ be defined by*

$$\begin{aligned}\Phi(A) = \;& 3\textstyle\sum_{i=1}^4 E_{ii}AE_{ii}^\dagger + E_{12}AE_{12}^\dagger + E_{23}AE_{23}^\dagger + E_{34}AE_{34}^\dagger + E_{41}AE_{41}^\dagger \\ & -(\textstyle\sum_{i=1}^4 E_{ii})A(\textstyle\sum_{i=1}^4 E_{ii})^\dagger,\end{aligned} \tag{7.1}$$

$$\begin{aligned}\Phi'(A) = \;& 3\textstyle\sum_{i=1}^4 E_{ii}AE_{ii}^\dagger + E_{13}AE_{13}^\dagger + E_{24}AE_{24}^\dagger + E_{31}AE_{31}^\dagger + E_{42}AE_{42}^\dagger \\ & -(\textstyle\sum_{i=1}^4 E_{ii})A(\textstyle\sum_{i=1}^4 E_{ii})^\dagger\end{aligned} \tag{7.1$'$}$$

and

$$\begin{aligned}\Phi''(A) = \;& 3\textstyle\sum_{i=1}^4 E_{ii}AE_{ii}^\dagger + E_{14}AE_{14}^\dagger + E_{21}AE_{21}^\dagger + E_{32}AE_{32}^\dagger + E_{43}AE_{43}^\dagger \\ & -(\textstyle\sum_{i=1}^4 E_{ii})A(\textstyle\sum_{i=1}^4 E_{ii})^\dagger\end{aligned} \tag{7.1$''$}$$

for every $A \in \mathcal{B}(H)$, where $E_{ji} = |j'\rangle\langle i|$. Then Φ, Φ', Φ'' are positive finite rank elementary operators that are not completely positive. Moreover, Φ and Φ'' are indecomposable.

Proof. Still, we only prove that Φ, Φ' and Φ'' are NCP positive. The fact that Φ and Φ'' are indecomposable will be illustrated by Example 7.2 or 7.3 below.

It is clear from Corollary 2.6 that Φ is not completely positive because $\sum_{i=1}^4 E_{ii}$ is not a contractive linear combination of $\{\sqrt{3}E_{11}, \ldots, \sqrt{3}E_{44}, E_{12}, E_{23}, E_{34}, E_{41}\}$. We will show that Φ is positive. Extend $\{|i\rangle\}_{i=1}^4$ and $\{|j'\rangle\}_{j=1}^4$ to orthonormal bases $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j'\rangle\}_{j=1}^{\dim K}$

of H and K, respectively. Then every $A \in \mathcal{B}(H)$ has a matrix representation $A = (a_{kl})$. Obviously, Φ maps $A = (a_{kl})$ to the matrix

$$\Phi(A) = \begin{pmatrix} 2a_{11}+a_{22} & -a_{12} & -a_{13} & -a_{14} & 0 & \cdots \\ -a_{21} & 2a_{22}+a_{33} & -a_{23} & -a_{24} & 0 & \cdots \\ -a_{31} & -a_{32} & 2a_{33}+a_{44} & -a_{34} & 0 & \cdots \\ -a_{41} & -a_{42} & -a_{43} & 2a_{44}+a_{11} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Take any unit vector $|x\rangle = (x_1, x_2, x_3, x_4, x_5, \cdots)^T \in H$ and consider the rank-one projection $|x\rangle\langle x|$. Obviously, Φ is positive if and only if $\Phi(|x\rangle\langle x|) \geq 0$ holds for all unit vector $|x\rangle \in H$. Since

$$\Phi(|x\rangle\langle x|) = \begin{pmatrix} 2|x_1|^2+|x_2|^2 & -x_1\bar{x_2} & -x_1\bar{x_3} & -x_1\bar{x_4} & 0 & \cdots \\ -x_2\bar{x_1} & 2|x_2|^2+|x_3|^2 & -x_2\bar{x_3} & -x_2\bar{x_4} & 0 & \cdots \\ -x_3\bar{x_1} & -x_3\bar{x_2} & 2|x_3|^2+|x_4|^2 & -x_3\bar{x_4} & 0 & \cdots \\ -x_4\bar{x_1} & -x_4\bar{x_2} & -x_4\bar{x_3} & 2|x_4|^2+|x_1|^2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

we see that $\Phi(|x\rangle\langle x|) \geq 0$ if and only if

$$M(x) = \begin{pmatrix} 2|x_1|^2+|x_2|^2 & -x_1\bar{x_2} & -x_1\bar{x_3} & -x_1\bar{x_4} \\ -x_2\bar{x_1} & 2|x_2|^2+|x_3|^2 & -x_2\bar{x_3} & -x_2\bar{x_4} \\ -x_3\bar{x_1} & -x_3\bar{x_2} & 2|x_3|^2+|x_4|^2 & -x_3\bar{x_4} \\ -x_4\bar{x_1} & -x_4\bar{x_2} & -x_4\bar{x_3} & 2|x_4|^2+|x_1|^2 \end{pmatrix} \geq 0.$$

It follows from Proposition 5.2 that all the principal minor determinants with order less than 4 of matrix $M(x)$ are semi-positive definite. So, to prove the positivity of $M(x)$, we need only to show that $\det(M(x)) \geq 0$. Writing $x_i = r_i e^{i\theta_i}$, $i = 1, 2, 3, 4$, we have

$$M(x) = U \begin{pmatrix} 2r_1^2+r_2^2 & -r_1r_2 & -r_1r_3 & -r_1r_4 \\ -r_1r_2 & 2r_2^2+r_3^2 & -r_2r_3 & -r_2r_4 \\ -r_1r_3 & -r_2r_3 & 2r_3^2+r_4^2 & -r_3r_4 \\ -r_1r_4 & -r_2r_4 & -r_3r_4 & 2r_4^2+r_1^2 \end{pmatrix} U^\dagger,$$

where

$$U = \begin{pmatrix} e^{i\theta_1} & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 \\ 0 & 0 & 0 & e^{i\theta_4} \end{pmatrix}$$

is a unitary matrix. It follows that Φ is positive if and only if the determinant

$$f(r_1,r_2,r_3,r_4)=\begin{vmatrix} 2r_1^2+r_2^2 & -r_1r_2 & -r_1r_3 & -r_1r_4 \\ -r_1r_2 & 2r_2^2+r_3^2 & -r_2r_3 & -r_2r_4 \\ -r_1r_3 & -r_2r_3 & 2r_3^2+r_4^2 & -r_3r_4 \\ -r_1r_4 & -r_2r_4 & -r_3r_4 & 2r_4^2+r_1^2 \end{vmatrix} \geq 0$$

holds for all $0 \leq r_1, r_2, r_3, r_4 \leq 1$ with $r_1^2+r_2^2+r_3^2+r_4^2=1$. This is the case since, by a computation, $\min f(r_1,r_2,r_3,r_4)=0$ (also, refer to the proof of Theorem 8.1). So Φ is positive, as desired.

Similarly, one can show that Φ' and Φ'' are positive but not completely positive. □

Now let us give some examples.

The entanglement of the states ρ in Example 7.2 were studied for 4×4 system in [13] by constructing suitable witnesses. We construct states ρ_t based on ρ and detect them by the positive maps obtained in Theorem 7.1. In addition, we also discuss the question when these states are entangled but cannot be recognized by the PPT criterion and the realignment criterion.

Example 7.2. Let H and K be Hilbert spaces of dimension ≥ 4, and let $\{|i\rangle\}_{i=1}^4$ and $\{|j'\rangle\}_{j=1}^4$ be any orthonormal sets of H and K, respectively. Let $|\omega\rangle = \frac{1}{2}(|11'\rangle + |22'\rangle + |33'\rangle + |44'\rangle)$. Define $\rho_1 = |\omega\rangle\langle\omega|$, $\rho_2 = \frac{1}{4}(|12'\rangle\langle 12'| + |23'\rangle\langle 23'| + |34'\rangle\langle 34'| + |41'\rangle\langle 41'|)$, $\rho_3 = \frac{1}{4}(|13'\rangle\langle 13'| + |24'\rangle\langle 24'| + |31'\rangle\langle 31'| + |42'\rangle\langle 42'|)$ and $\rho_4 = \frac{1}{4}(|14'\rangle\langle 14'| + |21'\rangle\langle 21'| + |32'\rangle\langle 32'| + |43'\rangle\langle 43'|)$. Let $\rho = \sum_{i=1}^4 q_i\rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i=1,2,3,4$ with $q_1+q_2+q_3+q_4=1$, $t\in[0,1]$, and ρ_0 is a state on $H\otimes K$. Then for sufficiently small t, or for ρ_0 with $(\Phi\otimes I)\rho_0 = (\Phi'\otimes I)\rho_0 = (\Phi''\otimes I)\rho_0 = 0$, the following statements are true.

(1) If $q_i < q_1$ for some $i=2,3,4$, then ρ_t is entangled.

(2) Let ρ_0 be PPT. Then ρ_t is PPT if and only if $q_2q_4 \geq q_1^2$ and $q_3^2 \geq q_1^2$. Thus, if $0<q_i<q_1<\frac{1}{4}$, $\frac{1}{4}\leq q_j<1$ with $q_iq_j\geq q_1^2$ and $0<q_1\leq q_3<1$, where $i,j\in\{2,4\}$ and $i\neq j$, then ρ_t is PPT entangled.

(3) If ρ_0 is PPT, and if $q_1\leq\frac{1}{7}$, $q_i=\frac{1}{2}q_1$, $q_j=\frac{1}{2}$ and $q_3=\frac{1}{2}-3q_i$, where $i,j\in\{2,4\}$ and $i\neq j$, then ρ_t is PPT entangled but can not be detected by the realignment criterion.

Like that in Example 6.3, we need only check ρ.

In the rest of this section, we will denote by $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j'\rangle\}_{j=1}^{\dim K}$ the orthonormal bases of H and K extended by $\{|i\rangle\}_{i=1}^4$ and $\{|j'\rangle\}_{j=1}^4$, respectively, denote by $F_{k,l}$ the rank one operator $|k'\rangle\langle l|$, which has a matrix representation of (k,l)-entry 1 and others 0 with respect to the above bases.

Thus, for $\rho=\sum_{i=1}^4 q_i\rho_i$, with respect to the above bases, we have

$$\begin{aligned}\rho = \; & \tfrac{1}{4}\mathrm{diag}(q_1,q_4,q_3,q_2,q_2,q_1,q_4,q_3,q_3,q_2,q_1,q_4,q_4,q_3,q_2,q_1) \\ & +\tfrac{q_1}{4}(F_{1,6}+F_{1,11}+F_{1,16}+F_{6,1}+F_{6,11}+F_{6,16} \\ & +F_{11,1}+F_{11,6}+F_{11,16}+F_{16,1}+F_{16,6}+F_{16,11})\end{aligned}$$

and

$$\begin{aligned}
& 4(\Phi\otimes I)(\rho) \\
= \; & \mathrm{diag}(2q_1+q_4, 2q_4+q_3, 2q_3+q_2, 2q_2+q_1, 2q_2+q_1, 2q_1+q_4, 2q_4+q_3, \\
& 2q_3+q_2, 2q_3+q_2, 2q_2+q_1, 2q_1+q_4, 2q_4+q_3, 2q_4+q_3, 2q_3+q_2, 2q_2+q_1, 2q_1+q_4) \\
& -q_1(F_{1,6}+F_{1,11}+F_{1,16}+F_{6,1}+F_{6,11}+F_{6,16} \\
& +F_{11,1}+F_{11,6}+F_{11,16}+F_{16,1}+F_{16,6}+F_{16,11}),
\end{aligned}$$

which is unitarily equivalent to

$$\begin{pmatrix} 2q_1+q_4 & -q_1 & -q_1 & -q_1 \\ -q_1 & 2q_1+q_4 & -q_1 & -q_1 \\ -q_1 & -q_1 & 2q_1+q_4 & -q_1 \\ -q_1 & -q_1 & -q_1 & 2q_1+q_4 \end{pmatrix} \oplus (2q_4+q_3)I_4$$
$$\oplus(2q_3+q_2)I_4 \oplus (2q_2+q_1)I_4 \oplus 0.$$

Hence, by Proposition 5.2, we get that $(\Phi\otimes I)(\rho) \not\geq 0$ if $q_4 < q_1$, which implies that ρ is entangled if $q_4 < q_1$.

Note that

$$\rho \text{ is PPT if and only if } q_2 q_4 \geq q_1^2 \text{ and } q_3 \geq q_1. \tag{7.2}$$

Thus we obtain that ρ is PPT entangled if $0 < q_4 < q_1 < \frac{1}{4}$, $\frac{1}{4} \leq q_2 < 1$ with $q_2q_4 \geq q_1^2$ and $0 < q_1 \leq q_3 < 1$. This reveals that the positive map Φ can recognize some PPT entangled states and hence is not decomposable.

The realignment matrix of ρ is

$$\begin{aligned}
\rho^R \cong \; & \tfrac{1}{4}\mathrm{diag}(q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1) \\
& +\tfrac{q_4}{4}(F_{1,6}+F_{6,11}+F_{11,16}+F_{16,1}) + \tfrac{q_3}{4}(F_{1,11}+F_{6,16}+F_{11,1}+F_{16,6}) \\
& +\tfrac{q_2}{4}(F_{1,16}+F_{6,1}+F_{11,6}+F_{16,11}) \\
\cong \; & \frac{1}{4}\begin{pmatrix} q_1 & q_4 & q_3 & q_2 \\ q_2 & q_1 & q_4 & q_3 \\ q_3 & q_2 & q_1 & q_4 \\ q_4 & q_3 & q_2 & q_1 \end{pmatrix} \oplus \tfrac{1}{4}q_1 I_{12} \oplus 0 = A \oplus \tfrac{1}{4}q_1 I_{12} \oplus 0.
\end{aligned}$$

Thus $\|\rho^R\|_1 = \|A\|_1 + 3q_1$. By computation, we have that

$$\begin{aligned}
\|A\|_1 = \; & \tfrac{3}{4}\sqrt{\textstyle\sum_{i=1}^4 q_i^2 - q_1q_2 - q_2q_3 - q_3q_4 - q_1q_4} \\
& +\tfrac{1}{4}\sqrt{\textstyle\sum_{i=1}^4 q_i^2 + 3(q_1q_2 + q_2q_3 + q_3q_4 + q_1q_4)}.
\end{aligned} \tag{7.3}$$

It follows from Eqs.(7.2)-(7.3) that the PPT criterion and the realignment criterion are independent each other. It is also easy to construct entangled states that can not be recognized by the PPT criterion and the realignment criterion. In fact, we have that $\|\rho^R\|_1 < 1$ if $q_1 \leq \frac{1}{7}$, $q_4 = \frac{1}{2}q_1$, $q_2 = \frac{1}{2}$ and $q_3 = \frac{1}{2} - 3q_4$. For example, $\|\rho^R\|_1 \doteq 0.9411 < 1$ if $q_1 = \frac{1}{7}$, $q_4 = \frac{1}{14}$, $q_2 = \frac{1}{2}$ and $q_3 = \frac{2}{7}$. Hence, in this case, the state ρ is PPT and cannot be detected by the realignment criterion. However it is entangled and can be recognized by the positive map Φ in Theorem 7.1.

Similarly, by applying the map Φ'', we have that ρ is entangled if $q_2 < q_1$, and, ρ is PPT entangled if $0 < q_2 < q_1 < \frac{1}{4}$, $\frac{1}{4} \leq q_4 < 1$ with $q_2 q_4 \geq q_1^2$ and $0 < q_1 \leq q_3 < 1$. Thus, Φ'' is indecomposable, too. Furthermore, if $q_1 \leq \frac{1}{7}$, $q_2 = \frac{1}{2}q_1$, $q_4 = \frac{1}{2}$ and $q_3 = \frac{1}{2} - 3q_2$, then ρ is PPT entangled that cannot be detected by the realignment criterion. However, it can be detected by the positive map Φ'' in Theorem 7.1.

By applying the map Φ', we see that ρ is entangled if $q_3 < q_1$. However, one should be careful that, in this case, ρ is not PPT. This means that we can not use ρ to check whether or not Φ' is decomposable.

Example 7.3. Let H and K be complex Hilbert spaces of dimension ≥ 4 and let $\{|i\rangle\}_{i=1}^4$ and $\{|j'\rangle\}_{j=1}^4$ be any orthonormal sets of H and K, respectively. Let

$$|\omega_1\rangle = \frac{1}{2}(|11'\rangle + |22'\rangle + |33'\rangle + |44'\rangle) \ \text{ and } \ |\omega_2\rangle = \frac{1}{2}(|12'\rangle + |23'\rangle + |34'\rangle + |41'\rangle).$$

Define $\rho_1 = |\omega_1\rangle\langle\omega_1|$, $\rho_2 = |\omega_2\rangle\langle\omega_2|$, $\rho_3 = \frac{1}{4}(|13'\rangle\langle 13'| + |24\rangle\langle 24'| + |31'\rangle\langle 31'| + |42'\rangle\langle 42'|)$ and $\rho_4 = \frac{1}{4}(|14'\rangle\langle 14'| + |21'\rangle\langle 21'| + |32'\rangle\langle 32'| + |43'\rangle\langle 43'|)$. Let $\rho = \sum_{i=1}^4 q_i\rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i = 1,2,3,4$ with $q_1 + q_2 + q_3 + q_4 = 1$, $t \in [0,1]$, and ρ_0 is a state on $H \otimes K$. By using of the positive finite rank elementary operators Φ, Φ' and Φ'' in Theorem 4.1, we get that, for sufficiently small t or for any ρ_0 with $(\Phi \otimes I)\rho_0 = (\Phi' \otimes I)\rho_0 = (\Phi'' \otimes I)\rho_0 = 0$, the followings are true.

(1) If $q_1 \neq q_2$ or $q_1 = q_2 > q_i$ for some $i \in \{3,4\}$, then ρ_t is entangled.

(2) Let ρ_0 be PPT. Then ρ_t is PPT if and only if $q_1(q_1q_3^2 - q_2^2q_3 - q_1^3) \geq q_2^2(q_1q_3 - q_2^2) \geq 0$ and $q_2(q_2q_4^2 - q_1^2q_4 - q_2^3) \geq q_1^2(q_2q_4 - q_1^2) \geq 0$. Hence, if, in addition, $q_1 \neq q_2$ or $q_1 = q_2 > q_i$ for some $i \in \{3,4\}$, then ρ_t is PPT entangled.

(3) If ρ_0 is separable, and if $\frac{1}{2}q_i = q_j \leq \frac{1}{15}$ and $q_3 = q_4$, where $i, j \in \{1,2\}$ and $i \neq j$, then ρ_t is PPT entangled that cannot be detected by the realignment criterion.

We need only deal with ρ.

For $\rho = q_1\rho_1 + q_2\rho_2 + q_3\rho_3 + 4\rho_4$, it is obvious that

$$\begin{aligned}\rho = \ & \tfrac{q_1}{4}(F_{1,1} + F_{1,6} + F_{1,11} + F_{1,16} + F_{6,1} + F_{6,6} + F_{6,11} + F_{6,16} \\ & + F_{11,1} + F_{11,6} + F_{11,11} + F_{11,16} + F_{16,1} + F_{16,6} + F_{16,11} + F_{16,16}) \\ & + \tfrac{q_2}{4}(F_{4,4} + F_{4,5} + F_{4,10} + F_{4,15} + F_{5,4} + F_{5,5} + F_{5,10} + F_{5,15} \\ & + F_{10,4} + F_{10,5} + F_{10,10} + F_{10,15} + F_{15,4} + F_{15,5} + F_{15,10} + F_{15,15}) \\ & + \tfrac{q_3}{4}(F_{3,3} + F_{8,8} + F_{9,9} + F_{14,14}) + \tfrac{q_4}{4}(F_{2,2} + F_{7,7} + F_{12,12} + F_{13,13}).\end{aligned}$$

Note that

$$\begin{aligned}4(\Phi \otimes I)(\rho) = \ & \mathrm{diag}(2q_1 + q_4, q_3 + 2q_4, q_2 + 2q_3, q_1 + 2q_2, q_1 + 2q_2, 2q_1 + q_4, q_3 + 2q_4, q_2 + 2q_3, \\ & q_2 + 2q_3, q_1 + 2q_2, 2q_1 + q_4, q_3 + 2q_4, q_3 + 2q_4, q_2 + 2q_3, q_1 + 2q_2, 2q_1 + q_4) \\ & - q_1(F_{1,6} + F_{1,11} + F_{1,16} + F_{6,1} + F_{6,11} + F_{6,16} \\ & + F_{11,1} + F_{11,6} + F_{11,16} + F_{16,1} + F_{16,6} + F_{16,11}) \\ & - q_2(F_{4,5} + F_{4,10} + F_{4,15} + F_{5,4} + F_{5,10} + F_{5,15} \\ & + F_{10,4} + F_{10,5} + F_{10,15} + F_{15,4} + F_{15,5} + F_{15,10}),\end{aligned}$$

which is unitarily equivalent to the operator $A \oplus B \oplus C \oplus D \oplus 0$, where

$$A = \begin{pmatrix} 2q_1+q_4 & -q_1 & -q_1 & -q_1 \\ -q_1 & 2q_1+q_4 & -q_1 & -q_1 \\ -q_1 & -q_1 & 2q_1+q_4 & -q_1 \\ -q_1 & -q_1 & -q_1 & 2q_1+q_4 \end{pmatrix}, B = \begin{pmatrix} q_1+2q_2 & -q_2 & -q_2 & -q_2 \\ -q_2 & q_1+2q_2 & -q_2 & -q_2 \\ -q_2 & -q_2 & q_1+2q_2 & -q_2 \\ -q_2 & -q_2 & -q_2 & q_1+2q_2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} q_2+2q_3 & 0 & 0 & 0 \\ 0 & q_2+2q_3 & 0 & 0 \\ 0 & 0 & q_2+2q_3 & 0 \\ 0 & 0 & 0 & q_2+2q_3 \end{pmatrix}, D = \begin{pmatrix} q_3+2q_4 & 0 & 0 & 0 \\ 0 & q_3+2q_4 & 0 & 0 \\ 0 & 0 & q_3+2q_4 & 0 \\ 0 & 0 & 0 & q_3+2q_4 \end{pmatrix}.$$

It is clear that $C, D \geq 0$. For the matrices A and B, by Proposition 5.2, we get that $A \geq 0$ if and only if $q_4 \geq q_1$ and $B \geq 0$ if and only if $q_1 \geq q_2$. So $(\Phi \otimes I)(\rho)$ is not positive if $q_4 < q_1$ or $q_1 < q_2$. It follows from the elementary operator criterion that ρ is entangled if $q_4 < q_1$ or $q_1 < q_2$.

Next, consider the positive partial transpose of ρ. It is clear that

$$\begin{aligned} \rho^{T_1} \cong\ & \tfrac{q_1}{4}(F_{1,1} + F_{2,5} + F_{3,9} + F_{4,13} + F_{5,2} + F_{6,6} + F_{7,10} + F_{8,14} \\ & + F_{9,3} + F_{10,7} + F_{11,11} + F_{12,15} + F_{13,4} + F_{14,8} + F_{15,12} + F_{16,16}) \\ & + \tfrac{q_2}{4}(F_{1,8} + F_{2,12} + F_{3,16} + F_{4,4} + F_{5,5} + F_{6,9} + F_{7,13} + F_{8,1} \\ & + F_{9,6} + F_{10,10} + F_{11,14} + F_{12,2} + F_{13,7} + F_{14,11} + F_{15,15} + F_{16,3}) \\ & + \tfrac{q_3}{4}(F_{3,3} + F_{8,8} + F_{9,9} + F_{14,14}) + \tfrac{q_4}{4}(F_{2,2} + F_{7,7} + F_{12,12} + F_{13,13}) \\ \cong\ & A_1 \oplus B_1 \oplus C_1 \oplus D_1 \oplus 0, \end{aligned}$$

where

$$A_1 = \frac{1}{4}\begin{pmatrix} q_1 & q_2 & 0 & 0 \\ q_2 & q_3 & 0 & q_1 \\ 0 & 0 & q_1 & q_2 \\ 0 & q_1 & q_2 & q_3 \end{pmatrix}, \quad B_1 = \frac{1}{4}\begin{pmatrix} q_4 & q_1 & q_2 & 0 \\ q_1 & q_2 & 0 & 0 \\ q_2 & 0 & q_4 & q_1 \\ 0 & 0 & q_1 & q_2 \end{pmatrix}$$

and

$$C_1 = \frac{1}{4}\begin{pmatrix} q_3 & 0 & q_1 & q_2 \\ 0 & q_1 & q_2 & 0 \\ q_1 & q_2 & q_3 & 0 \\ q_2 & 0 & 0 & q_1 \end{pmatrix}, \quad D_1 = \frac{1}{4}\begin{pmatrix} q_2 & 0 & 0 & q_1 \\ 0 & q_4 & q_1 & q_2 \\ 0 & q_1 & q_2 & 0 \\ q_1 & q_2 & 0 & q_4 \end{pmatrix}.$$

It is easy to check that $A_1 \geq 0$ if and only if $q_1q_3 \geq q_2^2$ and $q_1^2q_3^2 - 2q_1q_2^2q_3 - q_1^4 + q_2^4 \geq 0$; $B_1 \geq 0$ if and only if $q_2q_4 \geq q_1^2$ and $q_2^2q_4^2 - 2q_1^2q_2q_4 + q_1^4 - q_2^4 \geq 0$; $C_1 \geq 0$ if and only if $q_1q_3^2 \geq q_2^2q_3 + q_1^3$ and $q_1^2q_3^2 - 2q_1q_2^2q_3 - q_1^4 + q_2^4 \geq 0$; and $D_1 \geq 0$ if and only if $q_2q_4 \geq q_1^2$ and $q_2^2q_4^2 - 2q_1^2q_2q_4 + q_1^4 - q_2^4 \geq 0$. Hence

$$\begin{aligned} & \rho \text{ is PPT if and only if} \\ & q_1(q_1q_3^2 - q_2^2q_3 - q_1^3) \geq q_2^2(q_1q_3 - q_2^2) \geq 0 \\ \text{and}\quad & q_2(q_2q_4^2 - q_1^2q_4 - q_2^3) \geq q_1^2(q_2q_4 - q_1^2) \geq 0. \end{aligned} \tag{7.4}$$

Particularly,

$$\text{if } q_2 = 2q_1 \text{ and } \ q_3 = q_4 \geq 4q_1, \text{ then } \rho \ \text{ is PPT entangled.} \tag{7.5}$$

This fact will be used below.

Now, let us apply the realignment criterion to ρ. The realignment of ρ is

$$\begin{aligned}\rho^R \cong\ & \tfrac{1}{4}\mathrm{diag}(q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1)\\ & +\tfrac{q_2}{4}(F_{1,16} + F_{2,13} + F_{3,14} + F_{4,15} + F_{5,4} + F_{6,1} + F_{7,2} + F_{8,3}\\ & +F_{9,8} + F_{10,5} + F_{11,6} + F_{12,7} + F_{13,12} + F_{14,9} + F_{15,10} + F_{16,11})\\ & +\tfrac{q_3}{4}(F_{1,11} + F_{6,16} + F_{11,1} + F_{16,6}) + \tfrac{q_4}{4}(F_{1,6} + F_{6,11} + F_{11,16} + F_{16,1})\\ \cong\ & A \oplus B^{(3)} \oplus 0,\end{aligned}$$

where

$$A = \frac{1}{4}\begin{pmatrix} q_1 & q_4 & q_3 & q_2 \\ q_2 & q_1 & q_4 & q_3 \\ q_3 & q_2 & q_1 & q_4 \\ q_4 & q_3 & q_2 & q_1 \end{pmatrix}, \quad B = \frac{1}{4}\begin{pmatrix} q_1 & 0 & 0 & q_2 \\ q_2 & q_1 & 0 & 0 \\ 0 & q_2 & q_1 & 0 \\ 0 & 0 & q_2 & q_1 \end{pmatrix}$$

and $B^{(3)}$ denotes the direct sum of 3 copies of B. Then

$$\begin{aligned}\|\rho^R\|_1 =\ & \|A\|_1 + 3\|B\|_1\\ =\ & \tfrac{3}{4}\sqrt{\textstyle\sum_{i=1}^4 q_i^2 - q_1q_2 - q_2q_3 - q_3q_4 - q_1q_4}\\ & +\tfrac{1}{4}\sqrt{\textstyle\sum_{i=1}^4 q_i^2 + 3(q_1q_2 + q_2q_3 + q_3q_4 + q_1q_4)}\\ & +\tfrac{9}{4}\sqrt{q_1^2 + q_2^2 - q_1q_2} + \tfrac{3}{4}\sqrt{q_1^2 + q_2^2 + 3q_1q_2}.\end{aligned} \tag{7.6}$$

Now a computation reveals that, if $q_1 \leq \frac{1}{15}$, $q_2 = 2q_1$ and $q_3 = q_4$, then the trace norm $\|\rho^R\|_1 < 1$. Note that, by Eq.(7.5), ρ is PPT in this case. Hence, we get another kind of examples of entangled states that are PPT and cannot be detected by the realignment criterion.

Similarly, by using the positive map Φ'', we obtain that ρ is entangled if $q_2 < q_1$ or $q_3 < q_2$, and, if $q_2 \leq \frac{1}{15}$, $q_1 = 2q_2$ and $q_3 = q_4$, then ρ is PPT entangled that cannot be detected by the realignment criterion.

By using the positive map Φ', we see that ρ is entangled if $q_3 < q_1$ or $q_4 < q_2$. In this case, by Eq.(7.4), ρ is not PPT because $q_1q_3^2 - q_2^2q_3 - q_1^3 < 0$ or $q_2q_4^2 - q_1^2q_4 - q_2^3 < 0$.

8. Positive finite rank elementary operators of order (n, n)

In this section we consider the general case, that is, constructing positive finite rank elementary operators of order (n, n). The main purpose is to show that the following result is true.

Theorem 8.1. *Let H and K be Hilbert spaces of dimension $\geq n$, and let $\{|i\rangle\}_{i=1}^n$ and $\{|j'\rangle\}_{j=1}^n$ be any orthonormal sets of H and K, respectively. For $k = 1, 2, \cdots, n-1$, let $\Phi^{(k)} : \mathcal{B}(H) \to \mathcal{B}(K)$ be defined by*

$$\begin{aligned}\Phi^{(k)}(A) =\ & (n-1)\textstyle\sum_{i=1}^n E_{ii}AE_{ii}^\dagger + \sum_{i=1}^n E_{i,\pi^k(i)}AE_{i,\pi^k(i)}^\dagger\\ & -(\textstyle\sum_{i=1}^n E_{ii})A(\sum_{i=1}^n E_{ii})^\dagger\end{aligned} \tag{8.1}$$

for every $A \in \mathcal{B}(H)$, where $\pi(i) = \pi^1(i) = (i+1) \bmod n$, $\pi^k(i) = (i+k) \bmod n$ $(k > 1)$, $i = 1, 2, \cdots, n$ and $E_{ji} = |j'\rangle\langle i|$. Then $\Phi^{(k)}$ are positive but not completely positive. Moreover, $\Phi^{(k)}$ is indecomposable whenever either n is odd or $k \neq \frac{n}{2}$.

Proof. Obviously, $\Phi^{(k)}$ is not completely positive for each $k = 1, 2, \cdots, n-1$. Similar to the proof of Theorem 7.1, to prove that $\Phi = \Phi^{(1)}$ is positive, it is sufficient to show that the function

$$\begin{aligned} & f_{1,n}(r_1, r_2, \cdots, r_n) \\ = & \begin{vmatrix} (n-2)r_1^2 + r_2^2 & -r_1r_2 & -r_1r_3 & \cdots & -r_1r_n \\ -r_1r_2 & (n-2)r_2^2 + r_3^2 & -r_2r_3 & \cdots & -r_2r_n \\ -r_1r_3 & -r_2r_3 & (n-2)r_3^2 + r_4^2 & \cdots & -r_3r_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_1r_n & -r_2r_n & -r_3r_n & \cdots & (n-2)r_n^2 + r_1^2 \end{vmatrix} \\ \geq & \ 0 \end{aligned} \tag{7.2}$$

for all $(r_1, r_2, \cdots, r_n)$ with $0 \leq r_1, r_2, \cdots, r_n \leq 1$ and $\sum_{i=1}^n r_i^2 = 1$. Other $\Phi^{(k)}$s are dealt with similarly.

We may assume that all r_is are nonzero. Let $x_i = \frac{r_{i+1}^2}{r_i^2}$, $i = 1, 2, \ldots, n-1$, and $x_n = \frac{r_1^2}{r_n^2}$. Then $x_1x_2\cdots x_n = 1$ and

$$f_{1,n}(r_1, r_2, \cdots, r_n) = (r_1r_2\cdots r_n)^2 h_{1,n}(x_1, x_2, \cdots, x_n), \tag{7.3}$$

where

$$\begin{aligned} & h_{1,n}(x_1, x_2, \cdots, x_n) \\ = & \begin{vmatrix} (n-2) + x_1 & -1 & -1 & \cdots & -1 \\ -1 & (n-2) + x_2 & -1 & \cdots & -1 \\ -1 & -1 & (n-2) + x_3 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & (n-2) + x_n \end{vmatrix} \end{aligned} \tag{7.4}$$

with each $x_i > 0$ and $x_1x_2\cdots x_n = 1$. It follows that $f_{1,n} \geq 0$ for all $(r_1, r_2, \cdots, r_n)$ with $0 \leq r_1, r_2, \cdots, r_n \leq 1$ and $\sum_{i=1}^n r_i^2 = 1$ if and only if $h_{1,n} \geq 0$ holds for all $(x_1, x_2, \cdots, x_n)$ with $x_i > 0$ $(i = 1, 2, \ldots, n)$ and $x_1x_2\cdots x_n = 1$.

Note that, the determinant in Eq.(8.4) can be formulated as

$$\begin{aligned} h_{1,n}(x_1, x_2, \cdots, x_n) = \ & -M_0 + M_1 \textstyle\sum_{i=1}^n x_i + M_2 \sum_{i<j} x_ix_j + \cdots \\ & + M_k \textstyle\sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k} + \cdots \\ & + M_{n-1} \textstyle\sum_{i_1<i_2<\cdots<i_{n-1}} x_{i_1}x_{i_2}\cdots x_{i_{n-1}} + M_nx_1x_2\cdots x_n. \end{aligned}$$

The case of $n=3$ is obvious. So we assume that $n \geq 4$ in the sequel. Since, by Proposition 5.2, $h_{1,n}(0,0,\cdots,0) = -M_0 < 0$, we have $M_0 > 0$. Taking $x_i = 0$ for $2 \leq i \leq n$, we see that

$$-M_0 + M_1 x_1 = h_{1,n}(x_1, 0, \ldots, 0) = \begin{vmatrix} (n-2)+1 & -1 & -1 & \cdots & -1 \\ -1 & (n-2) & -1 & \cdots & -1 \\ -1 & -1 & (n-2) & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & (n-2) \end{vmatrix}$$

$$= \begin{vmatrix} (n-2) & -1 & -1 & \cdots & -1 \\ -1 & (n-2) & -1 & \cdots & -1 \\ -1 & -1 & (n-2) & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & (n-2) \end{vmatrix} + \begin{vmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & (n-2) & -1 & \cdots & -1 \\ 0 & -1 & (n-2) & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & (n-2) \end{vmatrix}$$

$$= \quad -M_0 + h_{1,n-1}(1,1,\ldots,1)x_1.$$

Thus we have $M_1 = h_{1,n-1}(1,1,\cdots,1) \geq 0$ by Proposition 5.2. Let $x_i = 0$ for $i \geq 3$. A computation reveals that $M_2 = h_{1,n-2}(2,2,\cdots,2) \geq 0$. In general, one can check that

$$M_k = h_{1,n-k}(k,k,\cdots k) \geq 0, \quad k = 1,2,\cdots,n. \tag{8.5}$$

For example,

$$\begin{aligned} M_{n-3} = \;& h_{1,3}(n-3,n-3,n-3) = \begin{vmatrix} n-2 & -1 & -1 \\ -1 & n-2 & -1 \\ -1 & -1 & n-2 \end{vmatrix} \\ = \;& (n-2)^3 - 3(n-2) - 2 \geq 0, \end{aligned}$$

$$M_{n-2} = h_{1,2}(n-2,n-2) = \begin{vmatrix} n-2 & -1 \\ -1 & n-2 \end{vmatrix} = (n-2)^2 - 1 \geq 0,$$

$M_{n-1} = h_{1,n-1} = n-2 \geq 0$ and $M_n = 1$. Thus we have shown that $M_0, M_1, M_2, \cdots M_n \in \mathbb{N} \cup \{0\}$. It is easily checked that $h_{1,n}(1,1,\cdots,1) = 0$. This leads to

$$\sum_{i=1}^{n} M_i = M_0. \tag{8.6}$$

Next, observe that if $a_j > 0$ and $a_1 a_2 \cdots a_m = 1$, then $\sum_{j=1}^m a_j \geq 1$. This fact implies that

$$\sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \geq 1 \tag{8.7}$$

holds for each $1 \leq k \leq n$. Eq.(8.7), together with Eq.(8.6), yields that $h_{1,n}(x_1, x_2, \cdots, x_n) \geq 0$ holds for all $(x_1, x_2, \cdots, x_n)$ with $x_1 x_2 \cdots x_n = 1$.

The last assertion will be proved by Example 8.4 below. The proof is finished. □

Remark 8.2. Let π be any permutation of $(1,2,\cdots,n)$ and let $\Psi_\pi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the map defined by

$$\Psi_\pi(A) = \operatorname{diag}\{(n-1)a_{11} + a_{\pi(1)\pi(1)}, (n-1)a_{22} + a_{\pi(2)\pi(2)}, \cdots, (n-1)a_{nn} + a_{\pi(n)\pi(n)}\} - A$$

for every $A = (a_{ij}) \in M_n(\mathbb{C})$. By Corollary 2.6, Proposition 5.3 and the proof of Theorem 8.1, it is easily seen that Ψ_π is a positive linear map that is not completely positive whenever $\pi \neq \mathrm{id}$.

Remark 8.3. For any n-dimensional Hilbert space H, define

$$D_k = \frac{1}{\sqrt{k(k+1)}}(\sum_{i=1}^{k-1} E_{ii} - (k-1)E_{kk}), \qquad k = 1, 2, \cdots, n-1,$$

$$M_{i,j} = \frac{1}{\sqrt{2}}(E_{ij} + E_{ji}) \ \text{ for } i < j,$$

and

$$N_{i,j} = \frac{1}{\sqrt{2}}(iE_{ij} - iE_{ji}) \ \text{ for } i < j.$$

Relabel these n^2-1 matrices as $J_1, J_2, \cdots, J_{n^2-1}$. Then the n^2-1 matrices form a completely orthonormal traceless set and any $n \times n$ Hermitian matrix S can be written as the form

$$S = \frac{1}{n}(I + \sum_{k=1}^{n^2-1} \eta_k J_k),$$

where $\eta_k \in \mathbb{R}$, $k = 1, 2, \cdots, n^2 - 1$. Hence it is clear that the $n \times n$ hermitian matrices with trace 1 and the points in $\mathbb{R}^{n^2-1}$ (the real linear space) are in one-to-one correspondence. The image Λ_n of the set of all density matrices is a closed convex set in $\mathbb{R}^{n^2-1}$. Then every positive linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ corresponds to a linear map $M_\Phi : \mathbb{R}^{n^2-1} \to \mathbb{R}^{n^2-1}$ that sends Λ_n into Λ_n. It was shown in [41] that every map represented by a matrix of the form $M = (n-1)^{-1}R$ is positive, where $R \in \mathcal{O}(n^2-1)$, the orthogonal group of proper and improper rotations in $\mathbb{R}^{n^2-1}$ ([41, Theorem 4]). Some more can be said. In fact, $M = (n-1)^{-1}R$ corresponds a positive map whenever $\|R\| \leq 1$. The positive maps in Theorem 6.2 may be obtained from this way. However, the positive maps in Theorem 7.1 can not be obtained from this way. For example, consider the map Φ in Theorem 7.1. By a simple calculation, we get

$$M_\Phi = \tfrac{1}{18}\begin{pmatrix} 9 & 3\sqrt{3} & 0 \\ -\sqrt{3} & 11 & 4\sqrt{2} \\ -2\sqrt{6} & -2\sqrt{2} & 10 \end{pmatrix}.$$

It is clear that $\|M_\Phi\| > \frac{1}{3}$, and so [41, Theorem 4] is not applicable to our map Φ here.

In the following we give two examples that generalize the examples in Sections 3-4.

The states ρ in Example 8.4 were suggested for $n \times n$ system in [13] without analyzing their entanglement.

Example 8.4. Let H and K be Hilbert spaces of dimension $\geq n$ and let $\{|i\rangle\}_{i=1}^n$ and $\{|j'\rangle\}_{j=1}^n$ be any orthonormal sets of H and K, respectively. Let $|\omega\rangle = \frac{1}{n}\sum_{i=1}^n |ii'\rangle$. Define $\rho_1 = |\omega\rangle\langle\omega|$, $\rho_2 = \frac{1}{n}\sum_{i=1}^n (I \otimes S)|ii'\rangle\langle ii'|(I \otimes S)^\dagger$, $\rho_3 = \frac{1}{n}\sum_{i=1}^n (I \otimes S^2)|ii'\rangle\langle ii'|(I \otimes S^2)^\dagger$, $\ldots$, $\rho_n = \frac{1}{n}\sum_{i=1}^n (I \otimes S^{n-1})|ii'\rangle\langle ii'|(I \otimes S^{(n-1)})^\dagger$, where S is the operator on K defined by $S|j'\rangle = |(j+1)'\rangle$ if $j = 1, 2, \cdots, n-1$, $S|n'\rangle = |1'\rangle$ and $S|j'\rangle = 0$ if $j > n$. Let $\rho = \sum_{i=1}^n q_i\rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n q_i = 1$, $t \in [0, 1]$, and ρ_0 is a state on $H\otimes K$. Then for sufficiently small t, or for ρ_0 with $(\Phi^{(k)}\otimes I)\rho_0 = 0$, $k = 1, 2, \cdots, n-1$, the following statements are true.

(1) If $q_i < q_1$ for some $i = 2, 3, \cdots, n$, then ρ_t is entangled.

(2) Let ρ_0 be PPT. Then ρ_t is a PPT state if and only if $q_i q_j \geq q_1^2$ for i, j with $i+j = n+2$, $i = 3, 4, \cdots, n$.

It is enough to discuss the entanglement of ρ. For $\rho = \sum_{i=1}^n q_i\rho_i$, by using of the map $\Phi = \Phi^{(1)}$ in Theorem 8.1, it is easily checked that

$$
\begin{aligned}
& n(\Phi\otimes I)(\rho) \\
\cong & \begin{pmatrix} (n-2)q_1+q_n & -q_1 & -q_1 & \cdots & -q_1 \\ -q_1 & (n-2)q_1+q_n & -q_1 & \cdots & -q_1 \\ -q_1 & -q_1 & (n-2)q_1+q_n & \cdots & -q_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_1 & -q_1 & -q_1 & \cdots & (n-2)q_1+q_n \end{pmatrix} \\
& \oplus((n-2)q_n+q_{n-1})I_n \oplus ((n-2)q_{n-1}+q_{n-2})I_n \oplus \cdots \oplus ((n-2)q_2+q_1)I_n \oplus 0.
\end{aligned}
$$

Thus, by Proposition 5.2, we get that ρ is entangled if $q_n < q_1$.

Similarly, by applying the map $\Phi^{(k)}$ in Theorem 8.1, we have ρ is entangled if $q_{n+1-k} < q_1$, where $k = 2, 3, \cdots, n-1$.

It is easily checked that ρ is PPT if and only if $q_i q_j \geq q_1^2$, where $i+j = n+2$ and $i = 3, 4, \cdots, n$.

Moreover, if n is odd, or if n is even but $k \neq \frac{n}{2}$, we can choose $q_1, q_2, \cdots q_n$ so that $q_{n+1-k} < q_1 < \frac{1}{n}$ and $q_i q_j \geq q_1^2$ whenever $i+j = n+2$. It follows that $\rho = \sum_{i=1}^n q_i\rho_i$ is PPT entangled which can be recognized by $\Phi^{(k)}$. Hence, $\Phi^{(k)}$ is not decomposable. This completes the proof of the last assertion of Theorem 8.1.

Example 8.5. Let H and K be complex Hilbert spaces of dimension $\geq n$ and let $\{|i\rangle\}_{i=1}^n$ and $\{|j'\rangle\}_{j=1}^n$ be any orthonormal sets of H and K, respectively. Let $|\omega_1\rangle = \frac{1}{\sqrt{n}}\sum_{i=1}^n |ii'\rangle$ and $|\omega_2\rangle = \frac{1}{\sqrt{n}}(|12'\rangle + |23'\rangle + \cdots + |(n-1)n'\rangle + |n1'\rangle)$. Define $\rho_1 = |\omega_1\rangle\langle\omega_1|$, $\rho_2 = |\omega_2\rangle\langle\omega_2|$, $\rho_3 = \frac{1}{n}\sum_{i=1}^n (I\otimes S^2)|ii'\rangle\langle ii'|(I\otimes S^{2\dagger})$, …, $\rho_n = \frac{1}{n}\sum_{i=1}^n (I\otimes S^{n-1})|ii'\rangle\langle ii'|(I\otimes S^{(n-1)\dagger})$, where S is the same operator as in Example 8.4. Let $\rho = \sum_{i=1}^n q_i\rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n q_i = 1$, $t \in [0, 1]$, and ρ_0 is a state on $H\otimes K$. By using of the positive finite rank elementary operators $\Phi^{(k)}$ in Theorem 8.1, one can get that, for sufficiently small t or for any ρ_0 with $(\Phi^{(k)}\otimes I)\rho_0 = 0$, $k = 1, 2, \cdots, n-1$, if $q_1 \neq q_2$ or $q_1 = q_2 > q_i$ for some $i \in \{3, 4, \cdots, n\}$, then ρ_t is entangled.

Still, we only need to consider the entanglement of ρ. For $\rho = \sum_{i=1}^{n} q_i \rho_i$, with $\Phi = \Phi^{(1)}$ as in Theorem 8.1, it is clear that

$$n(\Phi \otimes I)(\rho) \cong \begin{pmatrix} (n-2)q_1 + q_n & -q_1 & -q_1 & \cdots & -q_1 \\ -q_1 & (n-2)q_1 + q_n & -q_1 & \cdots & -q_1 \\ -q_1 & -q_1 & (n-2)q_1 + q_n & \cdots & -q_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_1 & -q_1 & -q_1 & \cdots & (n-2)q_1 + q_n \end{pmatrix}$$

$$\oplus \begin{pmatrix} (n-2)q_2 + q_1 & -q_2 & -q_2 & \cdots & -q_2 \\ -q_2 & (n-2)q_2 + q_1 & -q_2 & \cdots & -q_2 \\ -q_2 & -q_2 & (n-2)q_2 + q_1 & \cdots & -q_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_2 & -q_2 & -q_2 & \cdots & (n-2)q_2 + q_1 \end{pmatrix}$$

$$\bigoplus_{k=3}^{n} ((n-2)q_k + q_{k-1}) I_n \oplus 0$$

So, by Proposition 5.4, $(\Phi \otimes I)(\rho)$ is not positive if $q_n < q_1$ or $q_1 < q_2$, which implies that ρ is entangled if $q_n < q_1$ or $q_1 < q_2$.

Similarly, by applying the map $\Phi^{(k)}$ $(k = 2, 3, \cdots, n-1)$ in Theorem 8.1, one gets that ρ is entangled if $q_{n+1-k} < q_1$ or $q_1 < q_2$. Thus, we obtain that ρ is entangled if $q_1 \neq q_2$ or $q_1 = q_2 > q_i$ for some $i \in \{3, 4, \cdots, n\}$.

Before the end of this section, we propose a question.

Question 8.6. Let $n \geq 4$ be an even integer. Is the positive map $\Phi^{(\frac{n}{2})}$ defined in Theorem 8.1 indecomposable? Particularly, is the positive map Φ' defined in Theorem 7.1 indecomposable?

We guess that the answer is affirmative, but we are not able to prove it by now.

Acknowledgement. My research is partially supported by National Natural Science Foundation of China (10871111), Research Fund for the Doctoral Program of Higher Education of China (20101402110012).

References

[1] Augusiak R, Stasinska J, Phys. Rev. A 77 (2008) 010303.
[2] Bengtsson I, Zyczkowski K, Cambridge University Press, Cambridge, 2006.
[3] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A, Wootters W K, Phys. Rev. Lett. 70 (1993) 1895.
[4] Bouwmeester D, Pan J W, Mattle K, Eibl M, Weinfurter H, Zeilinger A, Nature 390 (1997) 575.
[5] D. Bruß, J. Math. Phys. 43 (2002) 4237.
[6] Cerf N J, Adami C, Gingrich R M, Phys. Rev. A 60 (1999): 893.
[7] Chen K, Wu L A, Quant. Inf. Comput 3 (2003) 193.
[8] Choi M D, Lin. Alg. Appl., 10(1975), 285-290.
[9] Choi M D, J. Operator Theory, 4(1980), 271-285.
[10] Choi M D, Proc. Sympos. Pure Math., 38(1982), 583-590.
[11] Chruściński D, Kossakowski A, Open Systems and Inf. Dynamics 14 (2007) 275;
[12] Chruściński D, Kossakowski A, J. Phys. A: Math. Theor. 41 (2008) 145301.

[13] M. A. Jafarizadeh, N. Behzadi, Y. Akbari, Eur. Phys. J. D 55 (2009) 197.

[14] Depillis J, Pacific J. Math., 23 (1967), 129-137.

[15] Deutsch D, Ekert A, Jozsa R, Macchiavello C, Popescu S, Sanpera A, Phys. Rev. Lett. 77 (1996) 2818.

[16] Deutsch D, Ekert A, Jozsa R, Macchiavello C, Popescu S, A. Sanpera, Phys. Rev. Lett. 80 (1998) 2022.

[17] Dixmier J, Von Neumann Algebras, North-Holland Publishing Com., Amsterdan, New York, Oxford, 1981.

[18] Douglas R G, Proc. Amer. Math. Soc. 17 (1966), 413-416.

[19] Effros E G, Ruan Z-J, Operator Spaces, Clarendon Press, Oxford, 2000.

[20] Grabowski J, Kuś M, Marmo G, Open. Sys. and Information Dyn. 14 (2007), 355-370.

[21] Guo Y and Hou J, The RCCN criterion of separability for states in infinite-dimensional quantum systems, pripringt.

[22] Horodecki M, Horodecki P, Horodecki R, Phys. Lett. A, 223 (1996), 1-8.

[23] Horodecki M, Horodecki P, Horodecki R, Phys. Lett. A 80 (1998) 5239.

[24] Horodecki M, Horodecki P, Phys. Rev. A, 59 (1999), 4206.

[25] Horodecki R, Horodecki P, Horodecki M, Rev. Mod. Phys. 81 (2009) 865.

[26] Hou J C, Sci. in China (ser.A), 32(1989), 929-940.

[27] Hou J C, Sci. in China (ser.A), 36(9) (1993), 1025-1035.

[28] Hou J C, J. Operator Theory, 39 (1998), 43-58.

[29] Hou J, J. Phys. A: Math. Theor. 43 (2010) 385201.

[30] Hou J C, Qi X F, Phys. Rev. A 81 (2010) 062351.

[31] Kadison R V, Ringrose J R, Fundamentals of the Theorey of Operator Algebras II, Graduate Studies in Math., 16, American Math. Society, New York: Academic Press, 1983.

[32] Kye S-H, Trends in Mathematics, Information Center for Mathematical Sciences, 6(2) (2003), 83-91.

[33] Mathieu M, Math. Ann., 284(1989), 223-244.

[34] Nielsen M A, Chuang I L, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, 2000.

[35] Paulsen V, Completely Bounded Maps and Operator Algebras, Cambridge Studies in Advanced Mathematics 78, Cambridge University Press, Cambridge, 2002.

[36] Peres A, Phys. Lett. A 202 (1996) 16.

[37] Qi X and Hou J, J. Phys. A: Math. Theor. 44 (2011) 215305

[38] Salgado D, Sánchez-Gómez J L, Open Systems and Information Dynamics, 12(1) (2005), 55-64.

[39] Shor P W, Phys. Rev. A 52 (1995) 2493.

[40] Skowronek Ł, Størmer E, Życzkowski K, J. Math. Phys. 50, 062106 (2009); doi:10.1063/1.3155378

[41] Simon S, S. P. Rajagopalan S P, Simon R, Pramana-Journal of Physics, 73(3) (2009) 471-483.

[42] Sperling J, Vogel W, Phys. Rev. A 79 (2009) 052313.

[43] Stinespring W F, Proc. Amer. Math. Soc., 6 (1955), 211-216.

[44] Størmer E, J. Func. Anal., 254 (2008), 2304-2313.

[45] Tóth G, Gühne O, Phys. Rev. Lett. 94 (2005) 060501.

[46] Werner R F, Phys. Rev. A 40 (1989) 4277.