Numerical Ranges of Weighted Shift Matrices

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$A \in M_n$

Definition (numerical range of A)

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$$

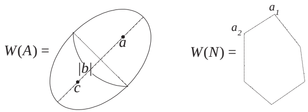
Note :

(1) $W(A) \subset \mathbb{C}$ is always compact and convex.

(2) $W(U^*AU) = W(A)$, where U is unitary.

(3) $\sigma(A) \subset W(A)$

(4) $N \in M_n$ is normal $\Rightarrow W(N) = \text{conv}\,(\sigma(N))$



(5) $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

(6) $A \cong A_1 \oplus \cdots \oplus A_k \Rightarrow W(A) = \text{conv}\,(W(A_1) \cup \cdots \cup W(A_k))$

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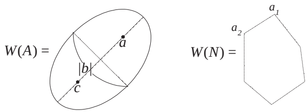

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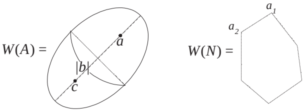

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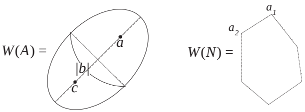

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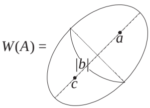

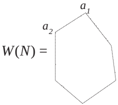

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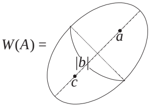

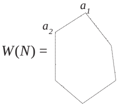

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$$A = \begin{bmatrix} 0 & a_1 & & \\ & 0 & \ddots & \\ & & \ddots & a_{n-1} \\ a_n & & & 0 \end{bmatrix}$$

A is called a weighted shift matrix with weights $a_1, \ldots, a_n$.

Note :

(1) $$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} A \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_2 & & \\ & 0 & \ddots & \\ & & \ddots & a_n \\ a_1 & & & 0 \end{bmatrix}$$

(2) $$\begin{bmatrix} 0 & a_1 & & \\ & 0 & \ddots & \\ & & \ddots & a_{n-1} \\ a_n & & & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & |a_1| & & \\ & 0 & \ddots & \\ & & \ddots & |a_{n-1}| \\ |a_n|e^{i\theta} & & & 0 \end{bmatrix}$$ where

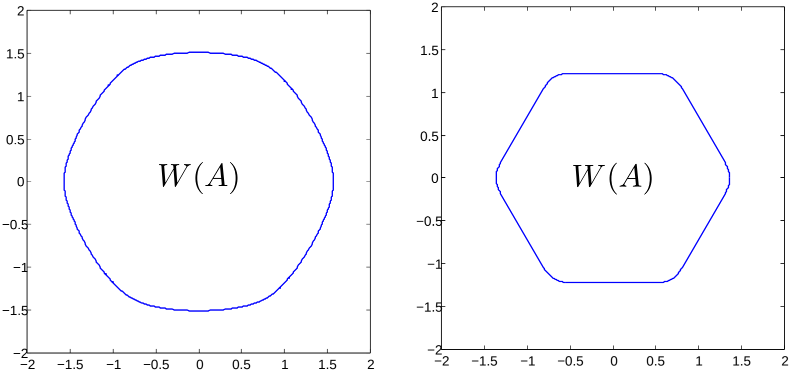
$\theta = \sum_{j=1}^{n} \arg(a_j)$

Hence, we have

$$A \cong \omega_n A, \ \omega_n = e^{2\pi i/n}.$$

That is,

$$W(A) = \omega_n W(A).$$

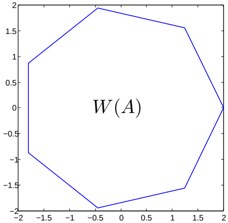


$n = 6$

If $|a_1| = |a_2| = \cdots = |a_n| \equiv r$.
Then

$$A \cong re^{i\theta} \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}.$$

In this case, A is normal.



$n = 7$

If $a_j = 0$, for some j.
Then

$$A \cong \begin{bmatrix} 0 & a_{j+1} & & & \\ & 0 & \ddots & & \\ & & & \ddots & a_{j-1} \\ 0 & & & & 0 \end{bmatrix}.$$

In this case,

$$W(A) = \{z \in \mathbb{C} : |z| \le r\}$$

where r is the maximal root of the equation

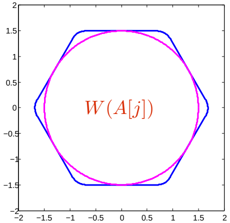
$$\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} S_l(|a_{j+1}|^2, \ldots, |a_{j-1}|^2, 0)(-\frac{1}{4})^l z^{n-2l} = 0.$$

In particular, $J_n = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \Rightarrow W(J_n) = \{z \in \mathbb{C} : |z| \le \cos \frac{\pi}{n+1}\}.$

$$A[1]=\begin{bmatrix} 0 & a_2 & & \\ & 0 & \ddots & \\ & & \ddots & a_{n-1} \\ 0 & & & 0 \end{bmatrix}, \ldots, A[j] \cong \begin{bmatrix} 0 & a_{j+1} & & \\ & 0 & \ddots & \\ & & \ddots & a_{j-2} \\ 0 & & & 0 \end{bmatrix}, \ldots$$

Now, if $\partial W(A)$ has a line segment on the line $x=r$, $r>0$.
Then r is the maximal eigenvalue of $\mathrm{Re}\, A$ with multiplicity at least two.
By the interlacing property, r is also the maximal eigenvalue of $\mathrm{Re}\, A[j]$ for all j.
That is,

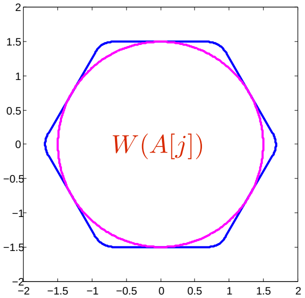
$$W(A[j]) = \{z\in\mathbb{C} : |z|\le r\} \text{ for all } j.$$



Theorem (Tsai & Wu, 2011)

$$A = \begin{bmatrix} 0 & a_1 & & & \\ & 0 & \ddots & & \\ & & & \ddots & a_{n-1} \\ a_n & & & & 0 \end{bmatrix}, \; a_j \neq 0 \text{ for all } j.$$

Then $\partial W(A)$ has a line segment if and only if $W(A[1]) = \cdots = W(A[n])$.



Lemma 1

If $w(A[j-1]) = w(A[j]) = w(A[j+1]) \equiv r$, then r is either the largest or the second largest eigenvalue of $\mathrm{Re}\,(e^{i\theta}A)$ for some θ.

Proof.
May assume $j = n$, $a_n < 0$ and $a_j > 0$ for $j = 1, \ldots, n-1$.
Compute the determinant of $(rI_n - \mathrm{Re}\,A)$ and obtain $\det(rI_n - \mathrm{Re}\,A) = 0$.

Lemma 2

If $w(A[1]) = \cdots = w(A[n]) \equiv r$, then r is a multiple eigenvalue of $\mathrm{Re}\,(e^{i\theta}A)$ for some θ.

Proof.
Let $p(z) = \det(zIn - \mathrm{Re}\,A)$, then $p'(r) = \sum_{n=1}^{n} \det(rI\,n-1 - \mathrm{Re}\,A[j]) = 0$.
Hence r is a multiple eigenvalue of $\mathrm{Re}\,A$.

Lemma 3

If the maximal eigenvalue r of $\mathrm{Re}\,A$ is multiple, then $\partial W(A)$ has a line segment on the line $x = r$

Proof. Take a vector $x \in \ker(rI_n - \mathrm{Re}\,A)$ such that $\langle \mathrm{Im}\,Ax, x\rangle \neq 0$.

Lemma 1'

If $w(A[j]) = w(A[k]) = w(A[l]) \equiv r$ for some $1 \leq j < k < l \leq n$, then r is either the largest or the second largest eigenvalue of $\mathrm{Re}\,(e^{i\theta}A)$ for some θ.

Proof.

May assume $a_n < 0$ and $a_j > 0$ for $j = 1, \ldots, n-1$.

Compute the determinant of $(rI_n - \mathrm{Re}\,A)$ and obtain $\det(rI_n - \mathrm{Re}\,A) = 0$.

Lemma 2'

If $w(A[j]) = w(A[k]) = w(A[l]) \equiv r$ for some $1 \leq j < k < l \leq n$, then r is a multiple eigenvalue of $\operatorname{Re}(e^{i\theta} A)$ for some θ.

Proof.

Assume $j = 1$, $l = n$ and $\operatorname{Re} A$ has eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$.

We want to show that $r = \lambda_1 = \lambda_2$.

If $r = \lambda_1$.

Take $x, y \in \mathbb{C}^{n-1}$ such that $(\operatorname{Re} A[1])x = rx$ and $(\operatorname{Re} A[n])y = ry$.
Let $x' = \begin{bmatrix} 0 \\ x \end{bmatrix} \in \mathbb{C}^n$ and $y' = \begin{bmatrix} y \\ 0 \end{bmatrix} \in \mathbb{C}^n$.
Then $x', y' \in \ker(\lambda_1 I_n - \operatorname{Re} A)$.

Since every entry of x and y is nonzero, hence $\dim \ker(\lambda_1 I_n - \operatorname{Re} A) \geq 2$.

Hence r is a multiple eigenvalue of $\operatorname{Re} A$.

Next, if $r = \lambda_2 \leq \lambda_1$.

Let u and v are unit eigenvectors of $\operatorname{Re} A$ w.r.t. λ_1 and λ_2, resp.,

$N = \text{span}\,\{u, v\}$. We have

$x' \equiv \begin{bmatrix} x \\ 0 \end{bmatrix} \in \left(\left(\mathbb{C}^{n-1} \oplus \{0\}\right) \cap N\right)$ and $y' \equiv \begin{bmatrix} 0 \\ y \end{bmatrix} \in \left(\left(\{0\} \oplus \mathbb{C}^{n-1}\right) \cap N\right)$.

Say $x' = au + bv$ and $y' = cu + dv$ with $|a|^2 + |b|^2 = 1 = |c|^2 + |d|^2$.

Then

$$r = w(A[n]) \geq \langle (\operatorname{Re}(A[n]))\, x, x\rangle = \left\langle (\operatorname{Re} A)\, x', x' \right\rangle = \lambda_1 |a|^2 + \lambda_2 |b|^2 \geq r|a|^2 + r|b|^2 = r,$$

the inequality here are actually equalities.

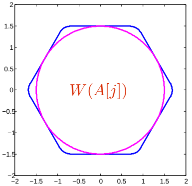
$\Rightarrow \lambda_1 = \lambda_2 = r$.

Theorem

$$A = \begin{bmatrix} 0 & a_1 & & & \\ & 0 & \ddots & & \\ & & & \ddots & a_{n-1} \\ a_n & & & & 0 \end{bmatrix}, a_j \neq 0 \text{ for all } j.$$

Then the following are equivalent:

1. $\partial W(A)$ has a line segment;
2. $W(A[1]) = \cdots = W(A[n])$;
3. $W(A[j]) = W(A[k]) = W(A[l])$ for some $1 \leq j < k < l \leq n$.



Theorem 1 (Tsai, 2011)

If A is an $n \times n$ weighted shift matrix with nonzero periodic weights $a_1, \ldots, a_k, a_1, \ldots, a_k, \ldots, a_1, \ldots, a_k$, $n = km$.
Then $\partial W(A)$ has a line segment.
Moreover,

$$W(A) = W(\begin{bmatrix} 0 & C & & \\ & 0 & \ddots & \\ & & \ddots & C \\ C & & & 0 \end{bmatrix}) = W(\begin{bmatrix} C & & & \\ & \omega_n C & & \\ & & \ddots & \\ & & & \omega_n^{m-1} C \end{bmatrix}),$$

where $C = \begin{bmatrix} 0 & a_1 & & \\ & 0 & \ddots & \\ & & \ddots & a_{k-1} \\ a_k & & & 0 \end{bmatrix}$.

Theorem 2 (Tsai, 2011)

$\partial W(A)$ contains a noncircular elliptic arc if and only if the aj's are nonzero, n is even, $|a_1| = |a_3| = \cdots = |a_{n-1}|$, $|a_2| = |a4| = \cdots = |a_n| and |a_1| \neq |a_2|$.

Question 1

If A is an $n \times n$ weighted shift matrix and $\partial W(A)$ has a line segment.
Are the weights periodic?

Ans. No!
We can find a 5×5 weighted shift matrix A such that $\partial W(A)$ has a line segment.

Question 2

Let A and B are $n \times n$ weighted shift matrices. If the weights of A are periodic and $W(A) = W(B)$.
Are the weights of B periodic?

Ans. I don't know!

Theorem

Let A and B are $n \times n$ ($n \geq 3$) weighted shift matrices with nonzero weights $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$. The following are equivalent :

1. $W(A) = W(B)$;
2. $p_A(x, y, z) = p_B(x, y, z)$;
3. $S_l(|a_1|^2, \ldots, |a_n|^2) = S_l(|b_1|^2, \ldots, |b_n|^2)$, for all $1 \leq l \leq \lfloor \frac{n}{2} \rfloor$ and $\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} b_i$.

Note.

1. $p_A(x, y, z) = \det(x\mathrm{Re}\, A + y\mathrm{Im}\, A + zI_n)$.
2. Our formulae involve the circularly symmetric function $S_r(a_1, \ldots a_n)$, where n and r are nonnegative integers. S_0 is define to be 1, while for $r \geq 1$, $S_r(a_1, \ldots a_n) = \sum \{\prod_{k=1}^{r} a_{\pi(k)} \,|\, \pi : (1, \ldots, r) \to (1, \ldots, n)$, where $\pi(k) + 1 < \pi(k+1)$ for $1 \leq k < r$, and if $\pi(1) = 1$ then $\pi(r) \neq n\}$.

Thank you for your attention!