Operator approach to Quantum Error Correction

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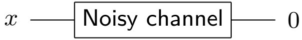
Outline

- Quantum Error Correction with syndrome measurement
- Quantum Error Correction without syndrome measurement
- Operator Quantum Error Correction
- Joint Rank-k Numerical Range
- Application on Fully Correlated Noise

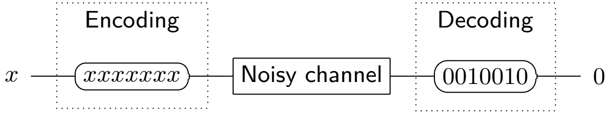
Classical error correction

- In classical conventional computer, data is stored and processed using binary bit $x \in \{0, 1\}$.
- Suppose in a noisy channel, each bit flips independent with a probability $p << 1$.
- Now a bit x is transmitted through the channel,



What is x?

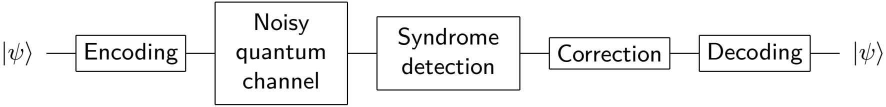
- Majority vote:



$x = 0$!

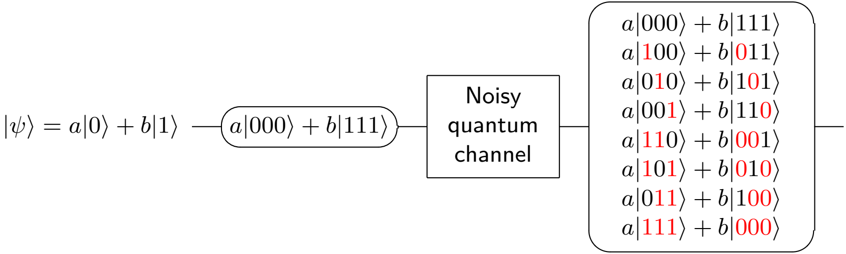
Quantum error correction with syndrome measurement

- Now suppose in a noisy quantum channel, each qubit flips independent with a probability $p << 1$.
- Due to the No-Cloning Theorem, the classical method cannot be applicable to qubits! i.e., $|\psi\rangle \not\to |\psi\rangle|\psi\rangle|\psi\rangle$.
- Quantum error correction with syndrome measurement:



Quantum error correction with syndrome measurement

Step 1: Encoding and transmission:



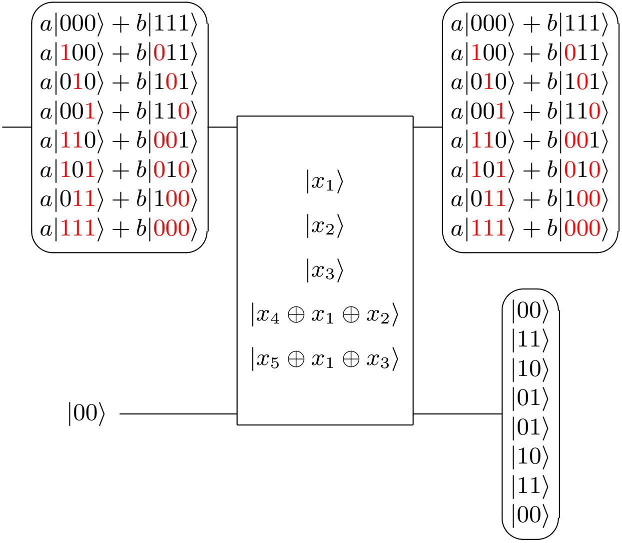
- $a|000\rangle + b|111\rangle \neq |\psi\rangle|\psi\rangle\psi\rangle$!
- Suppose U is a 8×8 unitary matrix such that

$$\begin{aligned} U|000\rangle &= |000\rangle \\ U|100\rangle &= |111\rangle \end{aligned}$$

Then the encoding can be regarded as

$$\begin{aligned} a|0\rangle + b|1\rangle &\longmapsto (a|0\rangle + b|1\rangle) \otimes |00\rangle = a|000\rangle + b|100\rangle \\ &\longmapsto U\,(a|000\rangle + b|100\rangle) = a|000\rangle + b|111\rangle \end{aligned}$$

Step 2: Syndrome detection:



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Quantum error correction with syndrome measurement

Step 3: Syndrome correction and decoding:

$$\begin{array}{c} a|000\rangle + b|111\rangle \\ a|100\rangle + b|011\rangle \\ a|010\rangle + b|101\rangle \\ a|001\rangle + b|110\rangle \\ a|110\rangle + b|001\rangle \\ a|101\rangle + b|010\rangle \\ a|011\rangle + b|100\rangle \\ a|111\rangle + b|000\rangle \end{array}$$

$$\begin{array}{c} |00\rangle \\ |11\rangle \\ |10\rangle \\ |01\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \\ |00\rangle \end{array}$$

$$\begin{array}{ll} |00\rangle : & I_2 \otimes I_2 \otimes I_2 \\ |01\rangle : & I_2 \otimes I_2 \otimes \sigma_x \\ |10\rangle : & I_2 \otimes \sigma_x \otimes I_2 \\ |11\rangle : & \sigma_x \otimes I_2 \otimes I_2 \end{array}$$

$$\begin{array}{c} a|000\rangle + b|111\rangle \\ a|000\rangle + b|111\rangle \\ a|000\rangle + b|111\rangle \\ a|000\rangle + b|111\rangle \\ a|111\rangle + b|000\rangle \\ a|111\rangle + b|000\rangle \\ a|111\rangle + b|000\rangle \\ a|111\rangle + b|000\rangle \end{array}$$

$$\begin{array}{c} a|0\rangle + b|1\rangle \\ a|1\rangle + b|0\rangle \end{array}$$

Important: The probability of error will be $p^2(3-2p) << p << 1$!

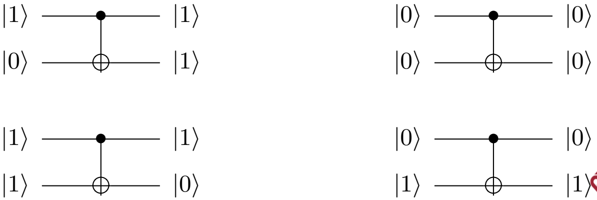
Gate

- A NOT gate acting on one qubit:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

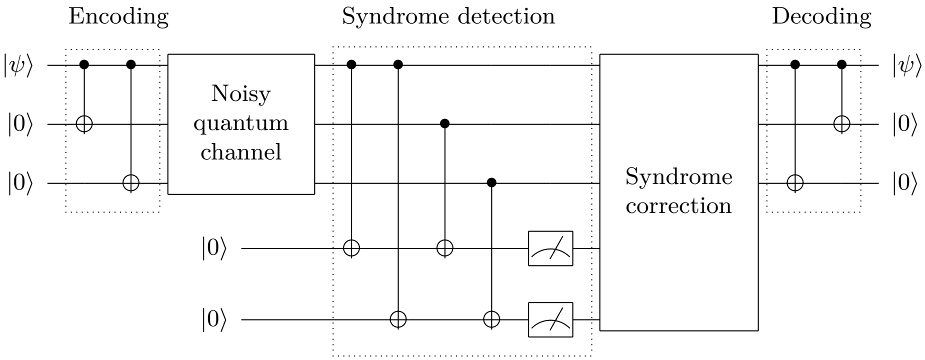
- A controlled-NOT (CNOT) gate acting on 2 qubits:

$$I_2 \oplus \sigma_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



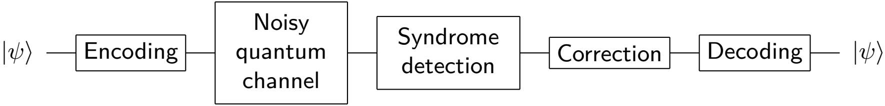
Quantum error correction with syndrome measurement

A circuit correcting error for bit-flip channel:

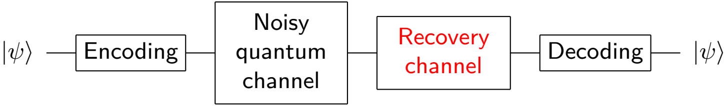


Operator Approach to Quantum Error Correction

- Quantum error correction with syndrome measurement:



- Quantum error correction without syndrome measurement:



Operator Approach to Quantum Error Correction

A quantum channel $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is a completely positive, trace preserving linear map of the form

$$\Phi : \rho \mapsto \sum_{j=1}^{r} E_j \rho E_j^\dagger \quad \text{with} \quad \sum_j E_j^\dagger E_j = I. \quad \text{[Choi (1975)]}$$

- Can one find another quantum channel $\Psi : B(\mathcal{H}) \to B(\mathcal{H})$ such that

$$\Psi \circ \Phi(\rho) = \rho \quad \text{for all} \quad P_{\mathcal{V}} \rho P_{\mathcal{V}} = \rho,$$

  where $P_{\mathcal{V}}$ is an orthogonal projection onto a k-dimensional subspace $\mathcal{V}$ of $\mathcal{H}$?
- If one write $P_{\mathcal{V}} = U(I_k \oplus 0)U^\dagger$ for some unitary U, then

$$P_{\mathcal{V}} \rho P_{\mathcal{V}} = \rho \quad \Longleftrightarrow \quad \rho = U \begin{bmatrix} \tilde{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger.$$

- The equation of recovery channel can be restated as

$$\Psi \circ \Phi \left(U \begin{bmatrix} \tilde{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \right) = U \begin{bmatrix} \tilde{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \quad \text{for all} \quad \tilde{\rho} \in M_k.$$

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- Recovery channel:

$$\Psi \circ \Phi \left(U \begin{bmatrix} \tilde{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \right) = U \begin{bmatrix} \tilde{\rho} & 0 \\ 0 & 0 \end{bmatrix} U^\dagger \quad \text{for all} \quad \tilde{\rho} \in M_k.$$

p-qubit data	Encoding to m-qubit	Noisy channel	Recovery channel	Decoding to p-qubit	
$\tilde{\rho}$ —	$\rho = U(\vert 0\rangle\langle 0\vert \otimes \tilde{\rho})U^\dagger$ —	Φ —	Ψ —	$\mathrm{tr}_1\left(U^\dagger \rho U\right)$ —	$\tilde{\rho}$

- If such k $(= 2^p)$-dimensional subspace $\mathcal{V}$ exists, $\mathcal{V}$ is called an quantum error correction code (QECC) for Φ (see Definition 1.3).
- When will such quantum error correction code exist??

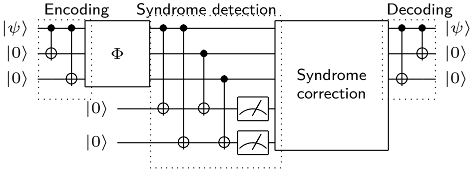
Theorem 1.5 - Existence of QECC [Knill, Laflamme (1996)]

A quantum channel $\Phi : \rho \mapsto \sum_{j=1}^{r} E_j \rho E_j^\dagger$ is correctable if and only if

$$P_{\mathcal{V}} E_i^\dagger E_j P_{\mathcal{V}} = \lambda_{ij} P_{\mathcal{V}} \quad \text{for all } 1 \le i, j \le r.$$

Example 1.7 Consider the three-qubit bit-flip channel $\Phi : M_8 \to M_8$ defined by

$$\Phi(\rho) = \sum_{j=0}^{3} X_j \rho X_j^\dagger,$$



with error operators

$$X_0 = \sqrt{p_0}\, I_2 \otimes I_2 \otimes I_2, \qquad X_1 = \sqrt{p_1}\, \sigma_x \otimes I_2 \otimes I_2,$$
$$X_2 = \sqrt{p_2}\, I_2 \otimes \sigma_x \otimes I_2, \qquad X_3 = \sqrt{p_3}\, I_2 \otimes I_2 \otimes \sigma_x,$$

where $\sum_{j=0}^{3} p_j = 1$.

Consider $\mathcal{V} = \text{span}\{|000\rangle, |111\rangle\}$ with orthogonal projection

$$P = |000\rangle\langle 000| + |111\rangle\langle 111| = E_{11} + E_{88}.$$

Following the proof of Knill-Laflamme result, one can construct the recovery channel as

$$\Psi(\rho) = P\rho P + (I - P)\rho(I - P).$$

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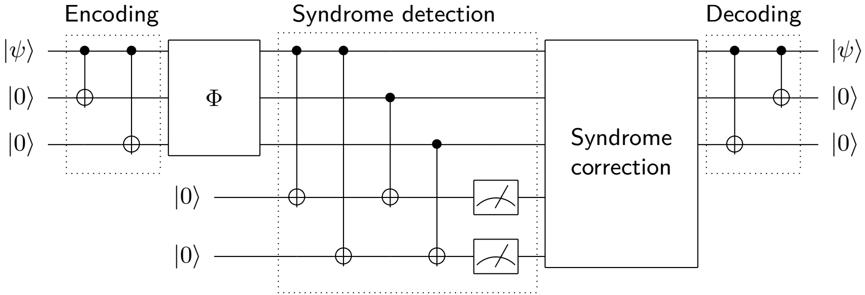
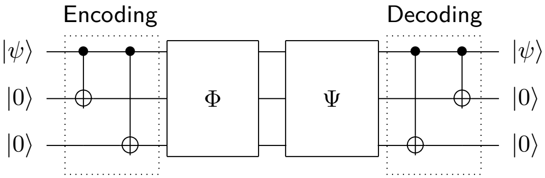
Quantum Error Correcting code

- With syndrome measurement:



- Without syndrome measurement:



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Remark 1.6

1. If we identify $\mathcal{H}$ with $\mathbb{C}^n$ and U is an $n \times n$ unitary matrix with columns $|u_1\rangle, \dots, |u_n\rangle$ so that the first k states $|u_1\rangle, \dots, |u_k\rangle$ form a basis for $\mathcal{V}$, where $k = \dim \mathcal{V}$, then condition (b) of Theorem 1.5 is equivalent to

$$U^\dagger E_i^\dagger E_j U = \begin{bmatrix} \lambda_{ij} I_k & * \\ * & * \end{bmatrix} \quad \text{for all} \quad 1 \le i, j \le r.$$

   This will lead to the discussion of joint higher rank numerical range later.

2. The proof of Theorem 1.5 is constructive and provides a procedure for constructing a recovery channel Ψ of Φ. However, the recovery channel Ψ may be hard to implement as the construction involves projection operators.

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Decoherence free subspace

Quantum error correcting code:

$$\Psi \circ \Phi(\rho) = \rho \quad \text{for all} \quad \rho \in B(\mathcal{H}) \text{ with } \rho = P_{\mathcal{V}} \rho P_{\mathcal{V}}.$$

p-qubit data	Encoding to m-qubit	Noisy channel	Recovery channel	Decoding to p-qubit	
$\tilde{\rho}$ —	$\rho = U(\|0\rangle\langle 0\| \otimes \tilde{\rho})U^\dagger$	Φ	Ψ	$\mathrm{tr}_1\left(U^\dagger \rho U\right)$	— $\tilde{\rho}$

Definition 1.8 - DFS

A subspace $\mathcal{V}$ of $\mathcal{H}$ is said to be a **decoherence free subspace (DFS)** for a quantum channel Φ on $B(\mathcal{H})$ if

$$\Phi(\rho) = \rho \quad \text{for all} \quad \rho \in B(\mathcal{H}) \text{ with } \rho = P_{\mathcal{V}} \rho P_{\mathcal{V}}, \tag{1}$$

where $P_{\mathcal{V}}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{V}$.

p-qubit data	Encoding to m-qubit	Noisy channel	Decoding to p-qubit	
$\tilde{\rho}$ —	$\rho = U(\|0\rangle\langle 0\| \otimes \tilde{\rho})U^\dagger$	Φ	$\mathrm{tr}_1\left(U^\dagger \rho U\right)$	— $\tilde{\rho}$

Notice that a decoherence free subspace is a QECC with $\Psi = id$.

Noiseless system

- For decoherence free subspace, the equation can be restated as

$$\Phi\left(U\begin{bmatrix}\tilde{\rho} & 0\\ 0 & 0\end{bmatrix}U^\dagger\right) = U\begin{bmatrix}\tilde{\rho} & 0\\ 0 & 0\end{bmatrix}U^\dagger \quad \text{for all} \quad \tilde{\rho}\in M_k.$$

- For any $\rho^A \in M_p$ and $\rho^B \in M_k$, there is a $\sigma^A \in M_p$ such that

$$\Phi\left(U\begin{bmatrix}\rho^A\otimes\rho^B & 0\\ 0 & 0\end{bmatrix}U^\dagger\right) = U\begin{bmatrix}\sigma^A\otimes\rho^B & 0\\ 0 & 0\end{bmatrix}U^\dagger.$$

Definition 1.10 - Noiseless subsystem

A subsystem $\mathcal{H}^B$ is said to be a **noiseless subsystem (NS)** for a quantum channel Φ on $B(\mathcal{H})$ if

1. $\mathcal{H}$ has a decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$; and
2. for any $\rho^A \in B(\mathcal{H}^A)$ and $\rho^B \in B(\mathcal{H}^B)$, there is $\sigma^A \in B(\mathcal{H}^A)$ such that

$$\Phi\left(\rho^A\otimes\rho^B\right) = \sigma^A\otimes\rho^B. \tag{2}$$

Noiseless system will reduce to decoherence free subspace if $\dim\mathcal{H}^A = 1$.

- QECC:

p-qubit data	Encoding to m-qubit	Noisy channel	Recovery channel	Decoding to p-qubit	
$\tilde{\rho}$	$\rho = U(\|0\rangle\langle 0\| \otimes \tilde{\rho})U^\dagger$	Φ	Ψ	$\mathrm{tr}_1\left(U^\dagger \rho U\right)$	$\tilde{\rho}$

- DFS:

p-qubit data	Encoding to m-qubit	Noisy channel	Decoding to p-qubit	
$\tilde{\rho}$	$\rho = U(\|0\rangle\langle 0\| \otimes \tilde{\rho})U^\dagger$	Φ	$\mathrm{tr}_1\left(U^\dagger \rho U\right)$	$\tilde{\rho}$

- NS:

p-qubit data	Encoding to m-qubit	Noisy channel	Decoding to p-qubit	
ρ^B	$\rho = U(\rho^A \otimes \rho^B)U^\dagger$	Φ	$\mathrm{tr}_A\left(U^\dagger \rho U\right)$	ρ^B

Example 1.12 Consider the quantum channel $\Phi : M_4 \to M_4$ with error operators $E_1 = F_1 \otimes I_2$ and $E_2 = F_2 \otimes I_2$, where

$$F_1 = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{1-\alpha} \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} 0 & \sqrt{\alpha} \\ \sqrt{1-\alpha} & 0 \end{bmatrix},$$

for some $0 \le \alpha \le 1$.

Decompose $\mathbb{C}^4 = \mathcal{H}^A \otimes \mathcal{H}^B$ with respect to the standard basis so that $\mathcal{H}^A = \mathcal{H}^B = \mathbb{C}^2$, i.e., $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$.

Then for any $\rho^A \in B(\mathcal{H}^A)$ and $\rho^B \in B(\mathcal{H}^B)$,

$$\begin{aligned} \Phi(\rho^A \otimes \rho^B) &= E_1(\rho^A \otimes \rho^B)E_1 + E_2(\rho^A \otimes \rho^B)E_2 \\ &= \left(F_1\rho^A F_1^\dagger + F_2\rho^A F_2^\dagger\right) \otimes \rho^B \\ &= \sigma^A \otimes \rho^B. \end{aligned}$$

Equivalent definitions for NS

Proposition 1.13 [Kribs et al (2006)]

Given a decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$ and a quantum channel Φ on $B(\mathcal{H})$. The following conditions are equivalent.

(1) $\mathcal{H}^B$ is a noiseless subsystem.

(2) For any $\rho^A \in B(\mathcal{H}^A)$ and $\rho^B \in B(\mathcal{H}^B)$, there is $\sigma^A \in B(\mathcal{H}^A)$ such that

$$\Phi(\rho^A \otimes \rho^B) = \sigma^A \otimes \rho^B.$$

(3) For any $\rho^B \in B(\mathcal{H}^B)$, there is $\sigma^A \in B(\mathcal{H}^A)$ such that

$$\Phi(I_A \otimes \rho^B) = \sigma^A \otimes \rho^B.$$

(4) For any $\rho^A \in B(\mathcal{H}^A)$ and $\rho^B \in B(\mathcal{H}^B)$,

$$\mathrm{tr}_A \left(P_{AB} \circ \Phi(\rho^A \otimes \rho^B)\right) = \rho^B,$$

where P_{AB} is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^A \otimes \mathcal{H}^B$.

Necessary and sufficient condition for existence of NS

Fixed orthonormal bases $\{|a_1\rangle, \ldots, |a_p\rangle\}$ and $\{|b_1\rangle, \ldots, |b_k\rangle\}$ for $\mathcal{H}^A$ and $\mathcal{H}^B$, respectively. Let

$$P_{ij} = |a_i\rangle\langle a_j| \otimes I_B \quad \text{for all} \quad 1 \leq i, j \leq p.$$

Notice that $P_{AB} = P_{11} + \cdots + P_{pp}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^A \otimes \mathcal{H}^B$.

Theorem 1.14 [Kribs. at el (2006)]

Given a decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$ and a quantum channel Φ on $B(\mathcal{H})$. Then $\mathcal{H}^B$ is a noiseless subsystem for Φ if and only if

$$E_s P_{AB} = P_{AB} E_s P_{AB} \quad \text{for all} \quad 1 \leq s \leq r, \tag{3}$$

and there are scalars $\lambda_{i,j,s} \in \mathbb{C}$ such that

$$P_{ii} E_s P_{jj} = \lambda_{i,j,s} P_{ij} \quad \text{for all} \quad 1 \leq i, j \leq p,\ 1 \leq s \leq r. \tag{4}$$

The equations (3) and (4) hold if and only if

$$U^\dagger E_s U = \begin{bmatrix} \Lambda^{(s)} \otimes I_B & * \\ 0 & * \end{bmatrix} \quad \text{with} \quad \Lambda^{(s)} = \left[\lambda_{i,j,s}\right] \quad \text{for all} \quad 1 \leq s \leq r.$$

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Recall that noiseless system will reduce to decoherence free subspace if $\mathcal{H}^A = 1$.

Corollary 1.16

Let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be a quantum channel. Then a subspace $\mathcal{V}$ of $\mathcal{H}$ is a decoherence free subspace for Φ if and only if there are scalars $\lambda_s \in \mathbb{C}$ such that

$$E_s P_{\mathcal{V}} = \lambda_s P_{\mathcal{V}} \quad \text{for all} \quad 1 \leq s \leq r. \tag{5}$$

The equation (5) hold if and only if

$$U^\dagger E_s U = \begin{bmatrix} \lambda_s I_B & * \\ 0 & * \end{bmatrix} \quad \text{for all} \quad 1 \leq s \leq r.$$

Example 1.17 Consider the quantum channel $\Phi : M_4 \to M_4$ with error operators

$$E_1 = \begin{bmatrix} \sqrt{1-2\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{1-2\alpha} \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} \sqrt{\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\alpha} \\ \sqrt{\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\alpha} \end{bmatrix}$$

for some $0 \leq \alpha \leq 1$. Let $U = E_{11} + E_{24} + E_{33} + E_{42}$. Then

$$U^\dagger E_1 U = \underbrace{\begin{bmatrix} \sqrt{1-2\alpha} & 0 \\ 0 & 1 \end{bmatrix}}_{\Lambda^{(1)}} \otimes I_2 \quad \text{and} \quad U^\dagger E_2 U = \underbrace{\begin{bmatrix} \sqrt{\alpha} & 0 \\ \sqrt{\alpha} & 0 \end{bmatrix}}_{\Lambda^{(2)}} \otimes I_2 .$$

Indeed, for any $\rho^B \in M_2$,

$$\Phi\left(U(I_A \otimes \rho^B)U^\dagger\right) = U(\sigma^A \otimes \rho^B)U^\dagger \quad \text{where} \quad \sigma^A = \begin{bmatrix} 1-\alpha & \alpha \\ \alpha & 1+\alpha \end{bmatrix} \quad \text{(Exercise!!)}$$

Equivalently, $\mathcal{H}^B$ is a noiseless subsystem if one decompose $\mathcal{H}$ to $\mathcal{H}^A \otimes \mathcal{H}^B$, $\dim \mathcal{H}^A = \dim \mathcal{H}^B = 2$, with respect to the basis $\{|00\rangle, |11\rangle, |10\rangle, |01\rangle\}$.

Exercise Show that this channel Φ has a 2-dimensional decoherence free subspace.

DFS vs NS vs QECC

- DFS:

$$\Phi\left(U\begin{bmatrix}\tilde\rho & 0\\ 0 & 0\end{bmatrix}U^\dagger\right)=U\begin{bmatrix}\tilde\rho & 0\\ 0 & 0\end{bmatrix}U^\dagger \quad \forall\tilde\rho\in M_k.$$

- NS: $\forall\rho^A\in M_p, \rho^B\in M_k,\ \exists\sigma^A\in M_p$ s.t.

$$\Phi\left(U\begin{bmatrix}\rho^A\otimes\rho^B & 0\\ 0 & 0\end{bmatrix}U^\dagger\right)=U\begin{bmatrix}\sigma^A\otimes\rho^B & 0\\ 0 & 0\end{bmatrix}U^\dagger.$$

- QECC:

$$\Psi\circ\Phi\left(U\begin{bmatrix}\tilde\rho & 0\\ 0 & 0\end{bmatrix}U^\dagger\right)=U\begin{bmatrix}\tilde\rho & 0\\ 0 & 0\end{bmatrix}U^\dagger \quad \forall\tilde\rho\in M_k.$$

- Under the QECC condition can we say something about Φ without the recovery channel Ψ? Yes!

$$\Phi\left(U\begin{bmatrix}E_{11}\otimes\tilde\rho & 0\\ 0 & 0\end{bmatrix}U^\dagger\right)=R\begin{bmatrix}\sigma\otimes\tilde\rho & 0\\ 0 & 0\end{bmatrix}R^\dagger \quad \forall\tilde\rho\in M_k.$$

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QECC Revisited

Theorem 1.18 [Li, Nakahara, Poon, Sze, Tomita (2011)]

Let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be a quantum channel with $n = \dim \mathcal{H}$. Suppose Φ has a k-dimensional quantum error correcting code $\mathcal{V}$ with orthogonal projection $P_{\mathcal{V}} = WW^\dagger$ with $W^\dagger W = I_k$. Then there is a unitary R and a positive definite $\sigma \in M_q$ with $q \le n/k$ such that

$$\Phi\left(W\tilde{\rho}W^\dagger\right) = R\begin{bmatrix}\sigma \otimes \tilde{\rho} & 0 \\ 0 & 0\end{bmatrix}R^\dagger \quad \text{for all} \quad \tilde{\rho} \in M_k.$$

In particular, if k divides n so that $B(\mathcal{H})$ can be regarded as $M_{n/k} \otimes M_k$, there is a positive semi-definite $\sigma \in M_{n/k}$ such that

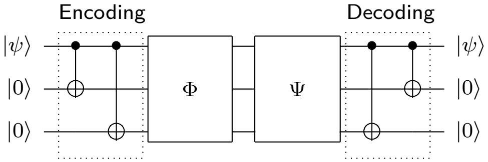
$$\Phi\left(W\tilde{\rho}W^\dagger\right) = R(\sigma \otimes \tilde{\rho})R^\dagger \quad \text{for all} \quad \tilde{\rho} \in M_k.$$

p-qubit data	Encoding to m-qubit	Noisy channel	Decoding to p-qubit	
ρ^B	$\rho = U(\lvert 0\rangle\langle 0\rvert \otimes \rho^B)U^\dagger$	Φ	$\mathrm{tr}_A\left(R^\dagger \rho R\right)$	ρ^B

QECC: bit-flip channel

Example 1.20 Consider the three-qubit bit-flip channel $\Phi : M_8 \to M_8$ defined by

$$\Phi(\rho) = \sum_{j=0}^{3} X_j \rho X_j^\dagger,$$



with error operators

$$X_0 = \sqrt{p_0}\, I_2 \otimes I_2 \otimes I_2, \qquad X_1 = \sqrt{p_1}\, \sigma_x \otimes I_2 \otimes I_2,$$

$$X_2 = \sqrt{p_2}\, I_2 \otimes \sigma_x \otimes I_2, \qquad X_3 = \sqrt{p_3}\, I_2 \otimes I_2 \otimes \sigma_x,$$

where $\sum_{j=0}^{3} p_j = 1$.

Consider $\mathcal{V} = \mathrm{span}\,\{|000\rangle, |111\rangle\}$. Following the proof of Theorem 1.18, one can construct the unitary matrices

$$U = E_{11} + E_{28} + E_{33} + E_{46} + E_{55} + E_{64} + E_{77} + E_{82}$$

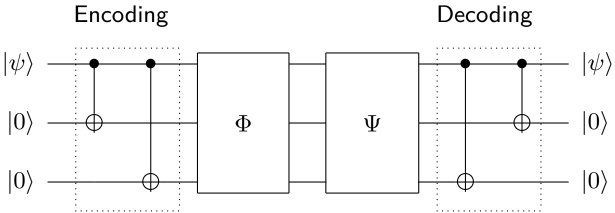
$$R = E_{11} + E_{27} + E_{35} + E_{44} + E_{53} + E_{66} + E_{78} + E_{82}.$$

Then there is $\sigma \in M_4$ such that

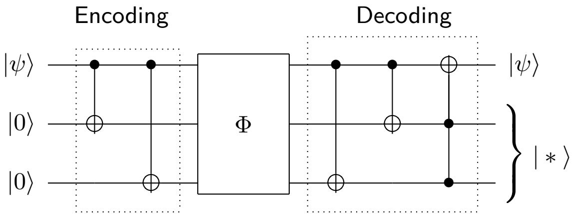
$$\Phi\left(U(|00\rangle\langle 00| \otimes \rho)U^\dagger\right) = R\left(\sigma \otimes \tilde{\rho}\right)R^\dagger \quad \text{for all} \quad \tilde{\rho} \in M_2.$$

QECC: bit-flip channel

- Original QECC:



- New QECC:



[Nakahara, Tomita (2011)]

Definition 1.21 - Correctable subsystem

A subsystem $\mathcal{H}^B$ is said to be a **correctable subsystem (CS)** for a quantum channel Φ on $B(\mathcal{H})$ if

1. $\mathcal{H}$ has a decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$, and
2. for any $\rho^A \in B(\mathcal{H}^A)$ and $\rho^B \in B(\mathcal{H}^B)$, there is $\sigma^A \in B(\mathcal{H}^A)$ such that

$$\Psi \circ \Phi(\rho^A \otimes \rho^B) = \sigma^A \otimes \rho^B. \tag{6}$$

Equivalently,

$$\mathrm{tr}_A \left(P_{AB} \circ \Psi \circ \Phi(\rho^A \otimes \rho^B)\right) = \rho^B \quad \text{for all} \quad \rho^A \in B(\mathcal{H}^A) \text{ and } \rho^B \in B(\mathcal{H}^B),$$

where P_{AB} is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^A \otimes \mathcal{H}^B$.

p-qubit data	Encoding to m-qubit	Noisy channel	Recovery channel	Decoding to p-qubit	
ρ^B	$\rho = U(\rho^A \otimes \rho^B)U^\dagger$	Φ	Ψ	$\mathrm{tr}_A \left(U^\dagger \rho U\right)$	ρ^B

Operator quantum error correction

A necessary and sufficient condition for the existence of correctable system was also given by Kribs et al.

Theorem 1.23 [Kribs et al. (2006)]

Given a decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$ and a quantum channel Φ on $B(\mathcal{H})$. Then $\mathcal{H}^B$ is a correctable subsystem for Φ if and only if there are scalars $\lambda_{i,j,s,t} \in \mathbb{C}$ such that

$$P_{ii} E_s^\dagger E_t P_{jj} = \lambda_{i,j,s,t}\, P_{ij} \quad \text{for all} \quad 1 \le i,j \le p,\ 1 \le s,t \le r. \tag{7}$$

The equation (7) holds if and only if there is a unitary U such that

$$U^\dagger E_s^\dagger E_t\, U = \begin{bmatrix} \Lambda^{(s,t)} \otimes I_B & * \\ * & * \end{bmatrix} \quad \text{with} \quad \Lambda^{(s,t)} = \left[\lambda_{i,j,s,t}\right] \quad \text{for all} \quad 1 \le s,t \le r.$$

Summary

DFS: $\forall\tilde{\rho}$ $\quad \Phi\left(U\begin{bmatrix}\tilde{\rho} & 0\\ 0 & 0\end{bmatrix}U^\dagger\right) = U\begin{bmatrix}\tilde{\rho} & 0\\ 0 & 0\end{bmatrix}U^\dagger$

NS: $\forall\rho^A,\rho^B,\exists\sigma^A$ $\quad \Phi\left(U\begin{bmatrix}\rho^A\otimes\rho^B & 0\\ 0 & 0\end{bmatrix}U^\dagger\right) = U\begin{bmatrix}\sigma^A\otimes\rho^B & 0\\ 0 & 0\end{bmatrix}U^\dagger$

QECC: $\forall\tilde{\rho}$ $\quad \Psi\circ\Phi\left(U\begin{bmatrix}\tilde{\rho} & 0\\ 0 & 0\end{bmatrix}U^\dagger\right) = U\begin{bmatrix}\tilde{\rho} & 0\\ 0 & 0\end{bmatrix}U^\dagger$

QECC: $\forall\tilde{\rho}$ $\quad \Phi\left(U\begin{bmatrix}E_{11}\otimes\tilde{\rho} & 0\\ 0 & 0\end{bmatrix}U^\dagger\right) = R\begin{bmatrix}\sigma\otimes\tilde{\rho} & 0\\ 0 & 0\end{bmatrix}R^\dagger$

OQEC: $\forall\rho^A,\rho^B,\exists\sigma^A$ $\quad \Psi\circ\Phi\left(U\begin{bmatrix}\rho^A\otimes\rho^B & 0\\ 0 & 0\end{bmatrix}U^\dagger\right) = U\begin{bmatrix}\sigma^A\otimes\rho^B & 0\\ 0 & 0\end{bmatrix}U^\dagger$

Summary

p-qubit data	Encoding to m-qubit	Noisy channel	Recovery channel	Decoding to p-qubit	
$\tilde{\rho}$	$\rho = U(\|0\rangle\langle 0\| \otimes \tilde{\rho})U^\dagger$	Φ		$\mathrm{tr}_1 \left(U^\dagger \rho U\right)$	$\tilde{\rho}$ **(DFS)**
ρ^B	$\rho = U(\rho^A \otimes \rho^B)U^\dagger$	Φ		$\mathrm{tr}_A \left(U^\dagger \rho U\right)$	ρ^B **(NS)**
$\tilde{\rho}$	$\rho = U(\|0\rangle\langle 0\| \otimes \tilde{\rho})U^\dagger$	Φ	Ψ	$\mathrm{tr}_1 \left(U^\dagger \rho U\right)$	$\tilde{\rho}$ **(QECC)**
$\tilde{\rho}$	$\rho = U(\|0\rangle\langle 0\| \otimes \rho^B)U^\dagger$	Φ		$\mathrm{tr}_A \left(R^\dagger \rho R\right)$	$\tilde{\rho}$ **(QECC)**
ρ^B	$\rho = U(\rho^A \otimes \rho^B)U^\dagger$	Φ	Ψ	$\mathrm{tr}_A \left(U^\dagger \rho U\right)$	ρ^B **(OQEC)**

Knill-Laflamme condition

Theorem 1.5 - Existence of QECC [Knill, Laflamme (1996)]

A quantum channel $\Phi : \rho \mapsto \sum_{j=1}^{r} E_j \rho E_j^\dagger$ is correctable if and only if

$$P_{\mathcal{V}} E_i^\dagger E_j P_{\mathcal{V}} = \lambda_{ij} P_{\mathcal{V}} \quad \text{for all } 1 \leq i, j \leq r.$$

Theorem 1.2

Suppose

$$\Phi(\rho) = \sum_{j=1}^{r} E_j \rho E_j^\dagger \quad \text{and} \quad \Psi(\rho) = \sum_{k=1}^{s} F_j \rho F_j^\dagger$$

are two quantum channels. By adding zero operators, if necessary, one can assume that $r = s$. Then $\Phi = \Psi$ if and only if there exists a $r \times r$ unitary matrix $U = [u_{ij}]$ such that

$$E_i = \sum_{j=1}^{r} u_{ij} F_j \quad \text{for all} \quad i = 1, \ldots, r.$$

Proof of the theorem can be found in [Nielsen & Chuang, Theorem 8.2].

Proof of Theorem 1.5

Suppose there is a recovery quantum channel $\Psi : B(\mathcal{H}) \to B(\mathcal{H})$ of the form $\Psi(\rho) = \sum_{k=1}^{p} R_k \rho R_k^\dagger$ such that

$$\Psi \circ \Phi(\rho) = \rho \quad \text{for all } \rho \text{ with } \rho = P\rho P.$$

Then

$$\sum_{k=1}^{p} \sum_{j=1}^{r} R_k E_j P \rho P E_j^\dagger R_k^\dagger = P\rho P \quad \text{for all} \quad \rho \in B(\mathcal{H}).$$

By Theorem 1.2, there are scalars $c_{jk} \in \mathbb{C}$ such that

$$R_k E_j P = c_{jk} P \quad \text{for all} \quad 1 \le j \le r,\ 1 \le k \le p.$$

Notice that $\sum_{k=1}^{p} R_k^\dagger R_k = I$. Thus for any $1 \le i, j \le r$,

$$P E_i^\dagger E_j P = \sum_{k=1}^{p} P E_i^\dagger R_k^\dagger R_k E_j P = \sum_{k=1}^{p} \overline{c}_{ik} c_{jk} P.$$

Then the condition holds with $\lambda_{ij} = \sum_{j=1}^{p} \overline{c}_{ik} c_{jk}$.

Proof of Theorem 1.5

Suppose that

$$PE_i^\dagger E_j P = \lambda_{ij} P \quad \text{for all} \quad 1 \le i, j \le r.$$

Let $\Lambda = [\lambda_{ij}]$.

Assumption: Λ is a $r \times r$ diagonal matrix with positive diagonal entires.

By polar decomposition, there is a unitary U_k such that

$$E_k P = U_k (PF_k^\dagger E_k P)^{\frac{1}{2}} = \sqrt{\lambda_{kk}}\, U_k P.$$

Let

$$P_k = U_k P U_k^\dagger = E_k P U_k^\dagger / \sqrt{\lambda_{kk}} \quad \text{for } k = 1, \ldots, r.$$

Then for any $1 \le k, \ell \le r$,

$$P_k^\dagger P_\ell = \begin{cases} U_k P U_k^\dagger & k = \ell, \\ 0 & k \neq \ell. \end{cases} \quad \Longrightarrow \quad U_k^\dagger P_k^\dagger P_\ell U_k = \begin{cases} P & k = \ell, \\ 0 & k \neq \ell. \end{cases}$$

Thus, the projections $P_1, \ldots, P_r$ are pairwise orthogonal.
Let

$$P_{r+1} = I - \sum_{k=1}^{r} P_k \quad \text{and} \quad U_{r+1} = I.$$

Notice that $P_{r+1}^2 = P_{r+1}$ and $P_{r+1}^\dagger P_j = 0$ for all $1 \le j \le r$.

Proof of Theorem 1.5

Define the recovery channel $\Psi : B(\mathcal{H}) \to B(\mathcal{H})$ by

$$\Psi(\rho) = \sum_{k=1}^{r+1} U_k^\dagger P_k \rho P_k U_k.$$

Clearly, $\sum_{k=1}^{r+1} P_k U_k U_k^\dagger P_k = \sum_{k=1}^{r+1} P_k = I$ and hence Ψ is trace preserving.

Notice that

$$\phi(\rho) = \sum_{k=1}^{r} E_k P \rho P E_k^\dagger = \sum_{k=1}^{r} \lambda_{kk} P_k U_k \rho U_k^\dagger P_k,$$

and so

$$\Psi \circ \Phi(\rho) = \sum_{\ell=1}^{r+1} \sum_{k=1}^{r} \lambda_{kk} \overbrace{U_\ell^\dagger P_\ell P_k U_k}^{P} \rho \overbrace{U_k^\dagger P_k P_\ell U_\ell}^{P} = \sum_{k=1}^{r} \lambda_{kk} P \rho P = P \rho P = \rho.$$

Thus, $\mathcal{V}$ is a quantum error correcting code for Φ.

Knill-Laflamme condition

Theorem 1.5 - Existence of QECC [Knill, Laflamme (1996)]

A quantum channel $\Phi : \rho \mapsto \sum_{j=1}^{r} E_j \rho E_j^\dagger$ is correctable if and only if

$$P_{\mathcal{V}} E_i^\dagger E_j P_{\mathcal{V}} = \lambda_{ij} P_{\mathcal{V}} \quad \text{for all } 1 \leq i, j \leq r.$$

The condition of Theorem 1.5 is equivalent to

$$U^\dagger E_i^\dagger E_j U = \begin{bmatrix} \lambda_{ij} I_k & * \\ * & * \end{bmatrix} \quad \text{for all} \quad 1 \leq i, j \leq r.$$

Joint rank-k numerical range

Choi, Kribs, and Życzkowski (2006) suggested the following:

Definition 2.1 - Joint rank-k numerical range

Given $A_1, \ldots, A_m \in M_n$. The (joint) rank-k numerical range $\Lambda_k(\mathbf{A})$ of the matrices $\mathbf{A} = (A_1, \ldots, A_m)$ is defined as the collection of $(a_1, \ldots, a_m) \in \mathbb{C}^{1\times m}$ such that

$$PA_jP = a_jP, \qquad j = 1, \ldots, m,$$

for some rank-k orthogonal projection P, i.e., That is,

$$\Lambda_k(\mathbf{A}) = \{(a_1, \ldots, a_m) \in \mathbb{C}^m : PA_jP = a_jP \text{ for some rank-}k\text{ orthogonal projection } P\}.$$

- A channel Φ has a k-dimensional correction code if and only if

$$\Lambda_k(E_1^\dagger E_1, E_1^\dagger E_2, \ldots, E_r^\dagger E_r) \neq \emptyset.$$

- Equivalently,

$$\Lambda_k(\mathcal{A}) = \{(a_1, \ldots, a_m) \in \mathbb{C}^m : X^\dagger A_j X = a_j I_k \text{ with } X^\dagger X = I_k\}.$$

- Also, $(a_1, \ldots, a_m) \in \Lambda_k(\mathcal{A})$ if and only if there is a unitary U such that

$$U^\dagger A_j U = \begin{bmatrix} a_j I_k & * \\ * & * \end{bmatrix} \quad \text{for } 1 \leq j \leq m.$$

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Rank-k numerical range

Example 2.2 A simple case. Given a bi-unitary channel

$$\Phi : \rho \mapsto tU_1 \rho U_1^\dagger + (1-t) U_2 \rho U_2^\dagger \quad \text{where } U_1 \text{ and } U_2 \text{ are unitary.}$$

The channel Φ is correctable if and only if

$$\Lambda_k(U_1^\dagger U_1, U_1^\dagger U_2, U_2^\dagger U_1, U_2^\dagger U_2) \neq \emptyset \quad \Longleftrightarrow \quad \Lambda_k(U_1^\dagger U_2) \neq \emptyset.$$

Rank-k numerical range

The rank-k numerical range of A on M_n is defined by

$$\Lambda_k(A) = \{\mu \in \mathbb{C} : PAP = \mu P \text{ for some rank-}k \text{ orthogonal projection } P\}.$$

- Equivalently,

$$\Lambda_k(A) = \{\mu \in \mathbb{C} : X^\dagger A X = \mu I_k \text{ with } X^\dagger X = I_k\}.$$

- For $k = 1$, it reduces to the **classical numerical range** defined as

$$W(A) = \{\langle x|A|x\rangle : |x\rangle \in \mathbb{C}^n \text{ with } \langle x|x\rangle = 1\}.$$

Rank-k numerical range

Basic properties of rank-k numerical range:

(P1) For any $a, b \in \mathbb{C}$, $\Lambda_k(aA + bI) = a\Lambda_k(A) + b$.

(P2) For any unitary $U \in M_n$, $\Lambda_k(U^\dagger AU) = \Lambda_k(A)$.

(P3) For any $n \times r$ matrix V with $r \geq k$ and $V^\dagger V = I_r$, we have $\Lambda_k(V^\dagger AV) \subseteq \Lambda_k(A)$.

(P4) Suppose $n < 2k$. The set $\Lambda_k(A)$ has at most one element.

(P5) $\Lambda_k(A)$ can be empty.

Example Let $A = \text{diag}\,(1, 1, 0, 0)$. Then $\Lambda_3(A) = \emptyset$.

Proof. Suppose $\Lambda_3(A) \neq \emptyset$. Then there is $U \in M_4$ such that

$$U^\dagger AU = \begin{bmatrix} \lambda I_3 & * \\ * & * \end{bmatrix}.$$

Then by interlacing inequality,

$$0 \leq \lambda \leq 0 \leq \lambda \leq 1 \leq \lambda \leq 1.$$

But this is impossible!

Rank-k numerical range

Theorem 2.3

Let $A \in M_n$ and $k \in \{1, \ldots, n\}$.

(a) If $n \geq 3k - 2$, then $\Lambda_k(A)$ is non-empty.

(b) If $n < 3k - 2$, there is $B \in M_n$ such that $\Lambda_k(B) = \emptyset$.

(c) If $A = A^\dagger$ has eigenvalues $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$, then

$$\Lambda_k(A) = [\lambda_{n-k+1}(A), \lambda_k(A)],$$

where the interval is an empty set if $\lambda_{n-k+1}(A) > \lambda_k(A)$ when $k > n/2$.

Theorem 2.3

(d) For any $A \in M_n$,

$$\Lambda_k(A) = \bigcap_{\xi \in [0,2\pi)} \left\{ \mu \in \mathbb{C} : e^{-i\xi}\mu + e^{i\xi}\bar{\mu} \leq \lambda_k(e^{-i\xi}A + e^{i\xi}A^\dagger) \right\},$$

where $\lambda_k(H)$ denotes the k-th largest eigenvalue of Hermitian $H \in M_n$.

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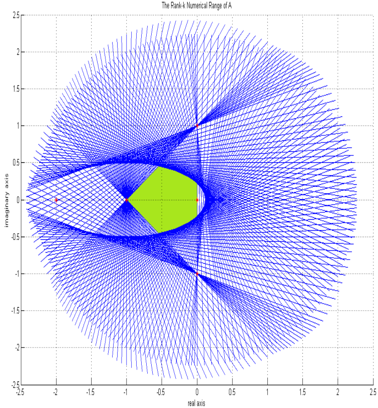
Example Let $A = \text{diag}\,(i, -i, -1) \oplus \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$\Lambda_k(A) = \bigcap_{\xi \in [0,2\pi)} \left\{ \mu \in \mathbb{C} : e^{-i\xi}\mu + e^{i\xi}\bar{\mu} \le \lambda_k(e^{-i\xi}A + e^{i\xi}A^\dagger) \right\},$$

The rank-2 numerical range of A is



Rank-k numerical range

Theorem 2.3

(d) For any $A \in M_n$,

$$\Lambda_k(A) = \bigcap_{\xi \in [0,2\pi)} \left\{ \mu \in \mathbb{C} : e^{-i\xi}\mu + e^{i\xi}\bar{\mu} \leq \lambda_k(e^{-i\xi}A + e^{i\xi}A^\dagger) \right\},$$

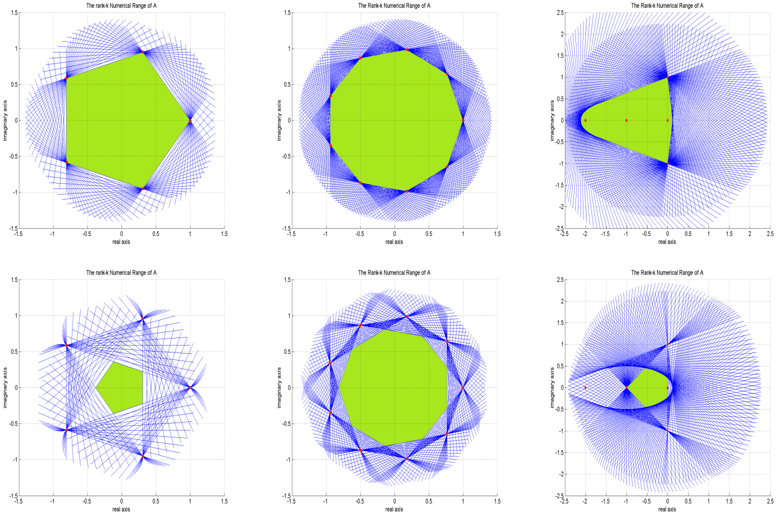
where $\lambda_k(H)$ denotes the k-th largest eigenvalue of Hermitian $H \in M_n$.

(e) $\Lambda_k(A)$ is always convex. [Woerdeman (2008)]

(f) If $A \in M_n$ is a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\Lambda_k(A) = \bigcap_{1 \leq j_1 < \cdots < j_{n-k+1} \leq n} \operatorname{conv}\{\lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}}\}.$$

Rank-1 and rank-2 numerical ranges of some matrices.



Joint rank-k numerical range

Recall that the joint rank-k numerical range of $\mathbf{A} = (A_1, \ldots, A_m)$ with A_j on M_n is defined by

$$\Lambda_k(\mathbf{A}) = \{(a_1, \ldots, a_m) \in \mathbb{C}^m : PA_jP = a_jP \text{ for some rank-}k\text{ orthogonal projection } P\}.$$

- Write $A_j = H_{2j-1} + iH_{2j}$ with Hermitian matrices

$$H_{2j-1} = \frac{1}{2}(A_j + A_j^\dagger) \quad \text{and} \quad H_{2j} = \frac{1}{2i}(A_j - A_j^\dagger).$$

- One can always identify

$$\begin{array}{ccc} \Lambda_k(A_1, \ldots, A_m) & \cong & \Lambda_k(H_1, H_2, \ldots, H_{2m-1}, H_{2m}) \\ \cap & & \cap \\ \mathbb{C}^m & & \mathbb{R}^{2m} \end{array}$$

- One can focus on $\Lambda_k(A_1, \ldots, A_m)$ with $A_1, \ldots, A_m$ Hermitian.
- In particular, $\Lambda_k(A_1 + iA_2) \cong \Lambda_k(A_1, A_2)$.

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Proposition 2.4

Suppose $\mathbf{A} = (A_1, \ldots, A_m) \in H_n^m$, and $T = [t_{ij}]$ is an $m \times r$ real matrix. If

$$B_j = \sum_{i=1}^{m} t_{ij} A_i \quad \text{for} \quad j = 1, \ldots, r,$$

and $\mathbf{B} = (B_1, \ldots, B_r)$, then

$$\{(a_1, \ldots, a_m)T : (a_1, \ldots, a_m) \in \Lambda_k(\mathbf{A})\} \subseteq \Lambda_k(\mathbf{B}).$$

The inclusion becomes equality if $\{A_1, \ldots, A_m\}$ is linearly independent and

$$\operatorname{span}\{A_1, \ldots, A_m\} = \operatorname{span}\{B_1, \ldots, B_r\}.$$

In view of the above proposition, in the study of the geometric properties of $\Lambda_k(\mathbf{A})$, we may always assume that $A_1, \ldots, A_m$ are linearly independent.

Joint rank-k numerical range

Proposition 2.5

Let $\mathbf{A} = (A_1, \ldots, A_m) \in H_n^m$, and let $k < n$.

(a) For any real vector $\mu = (\mu_1, \ldots, \mu_m)$,

$$\Lambda_k(A_1 - \mu_1 I, \ldots, A_m - \mu_m I) = \Lambda_k(\mathbf{A}) - \mu.$$

(b) If $(a_1, \ldots, a_m) \in \Lambda_k(\mathbf{A})$, then $(a_1, \ldots, a_{m-1}) \in \Lambda_k(A_1, \ldots, A_{m-1})$.

(c) $\Lambda_{k+1}(\mathbf{A}) \subseteq \Lambda_k(\mathbf{A})$.

(d) For any unitary $U \in M_n$,

$$\Lambda_k(U^\dagger A_1 U, \ldots, U^\dagger A_m U) = \Lambda_k(A_1, \ldots, A_m).$$

Non-emptyness

Question:
When will $\Lambda_k(\mathbf{A})$ be always non-empty for all Hermitian $\mathbf{A} = (A_1, \ldots, A_m)$?

Partial Answers:

1. $\Lambda_1(A_1, A_2, \ldots, A_m)$ is always non-empty.
2. If $n \geq 2k-1$, then $\Lambda_k(A_1) \neq \emptyset$. [Choi al et. (2006)]
3. If $n \geq 3k-2$, then

$$\Lambda_k(A_1, A_2) \equiv \Lambda_k(A_1 + iA_2) \neq \emptyset.$$

Proposition 2.6 [Knill, Laflamme, Viola (2000)]

Let $\mathbf{A} \in H_n^m$ and $1 < k < n$. Then $\Lambda_k(\mathbf{A})$ is non-empty if

$$n \geq (k-1)(m+1)^2.$$

However, the bound is not sharp.

When $(m,k) = (3,2)$

- Proposition 2.6:

$$\Lambda_2(A_1, A_2, A_3) \neq \emptyset \quad \text{if} \quad n \geq (k-1)(m+1)^2 = 16.$$

- It has been proved that

$$\Lambda_2(A_1, A_2, A_3) \neq \emptyset \quad \text{if} \quad n \geq 7$$

and

$$\Lambda_2(A_1, A_2, A_3) = \emptyset \quad \text{if} \quad n \leq 4.$$

Open problem

Is $\Lambda_2(A_1, A_2, A_3)$ always nonempty when $n = 5$ or 6?

Partial Answer:
Suppose $A_1, \ldots, A_m$ is a commuting family.

$$\Lambda_2(H_1, \ldots, H_m) \neq \emptyset \quad \text{if} \quad n \geq m + 2. \qquad \text{[Holbrook (2008)]}$$

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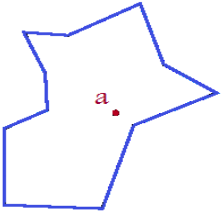
A set $S \subseteq \mathbb{R}^n$ is said to be star-shaped if there exists $a \in S$ such that

$$ta + (1-t)b \in S \quad \text{for all } b \in S \text{ and } 0 \le t \le 1.$$

The point a is called a star-center of S.



Theorem 2.9 [Li and Poon (2009)]

Given Hermitian $\mathbf{A} = (A_1, \ldots, A_m)$.

- If $\Lambda_\ell(\mathbf{A}) \neq \emptyset$ with $\ell \ge (m+2)k$ and $a \in \Lambda_\ell(\mathbf{A})$, then $\Lambda_k(\mathbf{A})$ is star-shaped with a as a star center.

In particular, when $n \ge 55$,

$$\Lambda_{10}(A_1, A_2, A_3) \neq \emptyset \quad \Longrightarrow \quad \Lambda_2(A_1, A_2, A_3) \text{ is star-shaped.}$$

Pauli matrices

- The Pauli matrices, also known as the spin matrices, and defined by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- Notice that

$$\begin{aligned} \sigma_x|0\rangle &= |1\rangle & \sigma_y|0\rangle &= i|1\rangle & \sigma_z|0\rangle &= |0\rangle \\ \sigma_x|1\rangle &= |0\rangle & \sigma_y|1\rangle &= -i|0\rangle & \sigma_z|1\rangle &= -|1\rangle \end{aligned}$$

- In general, for $|\psi\rangle = a|0\rangle + b|1\rangle$,

$$\begin{aligned} \sigma_x|\psi\rangle &= \sigma_x(a|0\rangle + b|1\rangle) = a|1\rangle + b|0\rangle \\ \sigma_y|\psi\rangle &= \sigma_y(a|0\rangle + b|1\rangle) = ia|1\rangle - ib|0\rangle \\ \sigma_z|\psi\rangle &= \sigma_z(a|0\rangle + b|1\rangle) = a|0\rangle - b|1\rangle \end{aligned}$$

- For any positive integer n, define

$$X_n = \sigma_x^{\otimes n}, \quad Y_n = \sigma_y^{\otimes n}, \quad \text{and} \quad Z_n = \sigma_z^{\otimes n}.$$

  Then

$$X_3|001\rangle = |110\rangle \qquad Y_3|001\rangle = i|110\rangle \qquad Z_3|001\rangle = -|001\rangle.$$

Fully correlated noise

- A noisy quantum channel is called fully correlated when all the qubits constituting the codeword are subject to the same error operators.
- This situation happens when size of the system is much smaller than the wavelength of the external disturbance causing the error.
- In general, such quantum channel has error operator of the form

$$W^{\otimes n} = W \otimes \cdots \otimes W \quad \text{with unitary } W \in M_2.$$

- Consider a fully correlated quantum channel $\Phi : M_{2^n} \to M_{2^n}$ of the form

$$\Phi(\rho) = p_0\rho + p_1 X_n \rho X_n^\dagger + p_2 Y_n \rho Y_n^\dagger + p_3 Z_n \rho Z_n^\dagger$$

with $p_0 + \cdots + p_4 = 1$.

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Fully correlated noise

By the Knill-Laflamme result, the fully correlated quantum channel $\Phi : M_{2^n} \to M_{2^n}$ by

$$\Phi(\rho) = p_0\rho + p_1 X_n \rho X_n^\dagger + p_2 Y_n \rho Y_n^\dagger + p_3 Z_n \rho Z_n^\dagger$$

has a k-dimensional quantum error correction code if and only if

$$\Lambda_k \begin{pmatrix} I_n & X_n & Y_n & Z_n \\ X_n^\dagger & X_n^\dagger X_n & X_n^\dagger Y_n & X_n^\dagger Z_n \\ Y_n^\dagger & Y_n^\dagger X_n & Y_n^\dagger Y_n & Y_n^\dagger Z_n \\ Z_n^\dagger & Z_n^\dagger X_n & Z_n^\dagger Y_n & Z_n^\dagger Z_n \end{pmatrix} \neq \emptyset.$$

As

$$\sigma_x\sigma_y = i\sigma_z, \quad \sigma_y\sigma_z = i\sigma_x, \quad \text{and} \quad \sigma_z\sigma_x = i\sigma_y,$$

it follows that

$$X_n^\dagger Y_n = i^n Z_n, \quad Y_n^\dagger Z_n = i^n X_n, \quad \text{and} \quad Z_n^\dagger X_n = i^n X_n.$$

It follows that QECC exists if and only if

$$\Lambda_k(X_n, Y_n, Z_n) \neq \emptyset.$$

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Theorem 3.1

Suppose $n > 2$ is odd. Then $\Lambda_{2^{n-1}}(X_n, Y_n, Z_n) \neq \emptyset$.

Indeed, $(0, 0, 1) \in \Lambda_{2^{n-1}}(X_n, Y_n, Z_n)$. (Exercise!)

By Theorem 1.18,

Theorem 3.2

Suppose n is odd and $\Phi : M_{2^n} \to M_{2^n}$ is a fully correlated quantum channel given by

$$\Phi(\rho) = p_0\rho + p_1 X_n \rho X_n^\dagger + p_2 Y_n \rho Y_n^\dagger + p_3 Z_n \rho Z_n^\dagger.$$

There exist a unitary $R \in M_{2^n}$ and a density matrix $\rho_a \in M_2$ such that

$$\Phi\left(R(|0\rangle\langle 0| \otimes \tilde{\rho})R^\dagger\right) = R\left(\rho_a \otimes \tilde{\rho}\right)R^\dagger \quad \text{for all} \quad \tilde{\rho} \in M_{2^{n-1}}.$$

So one can encode $(n-1)$-data qubit states to n-qubit codewords.

The unitary matrix R can be constructed explicitly.

When $n = 3$

For the quantum channel $\Phi : M_8 \to M_8$ given by

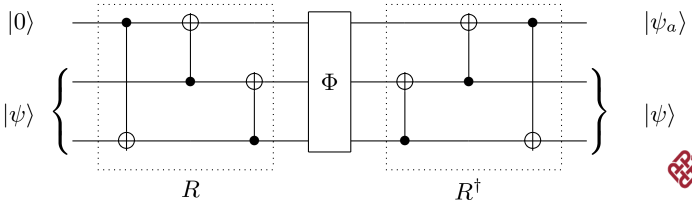
$$\Phi(\rho) = p_0\rho + p_1 X_3 \rho X_3^\dagger + p_2 Y_3 \rho Y_3^\dagger + p_3 Z_3 \rho Z_3^\dagger,$$

then

$$R^\dagger \Phi\left(R(|0\rangle\langle 0| \otimes \tilde{\rho})R^\dagger\right) R = \rho_a \otimes \tilde{\rho} \quad \text{for all} \quad \tilde{\rho} \in M_4,$$

where

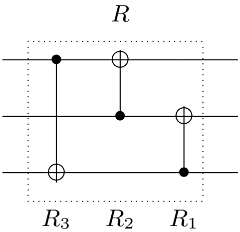
$$\begin{aligned} R &= E_{11} + E_{42} + E_{73} + E_{64} + E_{85} + E_{56} + E_{27} + E_{38} \\ &= |000\rangle\langle 000| + |011\rangle\langle 001| + |110\rangle\langle 010| + |101\rangle\langle 011| \\ &\quad +|111\rangle\langle 100| + |100\rangle\langle 101| + |001\rangle\langle 110| + |010\rangle\langle 111|. \end{aligned}$$



When $n = 3$

$$R = E_{11} + E_{42} + E_{73} + E_{64} + E_{85} + E_{56} + E_{27} + E_{38} = R_1 R_2 R_3$$

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



$$R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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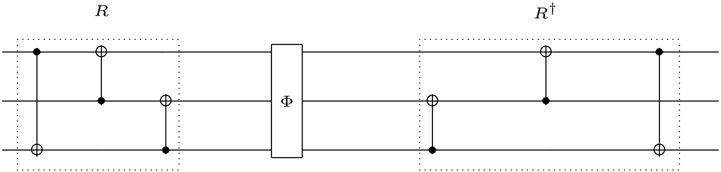
When $n = 3$



$|0\rangle \otimes |01\rangle$ → $|001\rangle$ → $|001\rangle$ → $|011\rangle$ →

I :	$\|011\rangle$	$\|001\rangle$	$\|001\rangle$	$\|001\rangle$	$\|0\rangle \otimes \|01\rangle$
X_3 :	$\|100\rangle$	$\|100\rangle$	$\|100\rangle$	$\|101\rangle$	$\|1\rangle \otimes \|01\rangle$
Y_3 :	$-i\|100\rangle$	$-i\|100\rangle$	$-i\|100\rangle$	$-i\|101\rangle$	$-i\|1\rangle \otimes \|01\rangle$
Z_3 :	$\|011\rangle$	$\|001\rangle$	$\|001\rangle$	$\|001\rangle$	$\|0\rangle \otimes \|01\rangle$

$|0\rangle \otimes |11\rangle$ → $|011\rangle$ → $|111\rangle$ → $|101\rangle$ →

I :	$\|101\rangle$	$\|111\rangle$	$\|011\rangle$	$\|011\rangle$	$\|0\rangle \otimes \|11\rangle$
X_3 :	$\|010\rangle$	$\|010\rangle$	$\|110\rangle$	$\|111\rangle$	$\|1\rangle \otimes \|11\rangle$
Y_3 :	$-i\|010\rangle$	$-i\|010\rangle$	$-i\|110\rangle$	$-i\|111\rangle$	$-i\|1\rangle \otimes \|11\rangle$
Z_3 :	$\|101\rangle$	$\|111\rangle$	$\|011\rangle$	$\|011\rangle$	$\|0\rangle \otimes \|11\rangle$

When $n = 5$

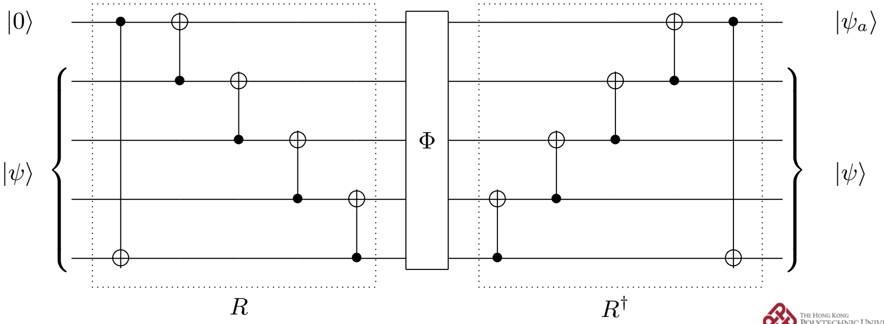
For the quantum channel $\Phi : M_{32} \to M_{32}$ given by

$$\Phi(\rho) = p_0\rho + p_1 X_5 \rho X_5^\dagger + p_2 Y_5 \rho Y_5^\dagger + p_3 Z_5 \rho Z_5^\dagger,$$

then

$$R^\dagger \Phi\left(R(|0\rangle\langle 0| \otimes \tilde{\rho})R^\dagger\right) R = \rho_a \otimes \tilde{\rho} \quad \text{for all} \quad \tilde{\rho} \in M_{16},$$

where R is a unitary matrix constructed by the following circuit.



Fully correlated noise

Theorem 3.4

Suppose $n > 2$ is even. Then

1. $\Lambda_{2n-2}(X_n, Y_n, Z_n) \neq \emptyset$.
2. $\Lambda_{2n-1}(X_n, Y_n, Z_n) = \emptyset$.

In this case, $(1, 1, 1) \in \Lambda_{2n-2}(X_n, Y_n, Z_n)$.

Theorem 3.5

Suppose n is even and $\Phi : M_{2^n} \to M_{2^n}$ is a fully correlated quantum channel given by

$$\Phi(\rho) = p_0\rho + p_1 X_n \rho X_n^\dagger + p_2 Y_n \rho Y_n^\dagger + p_3 Z_n \rho Z_n^\dagger.$$

There exists a unitary $R \in M_{2^n}$ such that

$$\Phi\left(R(|00\rangle\langle 00| \otimes \tilde{\rho})\, R^\dagger\right) = R\left(|00\rangle\langle 00| \otimes \tilde{\rho}\right) R^\dagger \quad \text{for all} \quad \tilde{\rho} \in M_{2^{n-2}}.$$

The output density matrix is the same as the input.

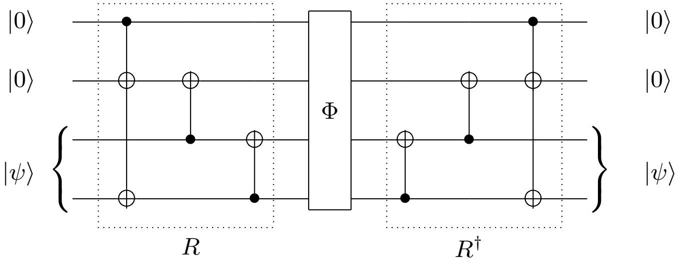
For the quantum channel $\Phi : M_{16} \to M_{16}$ given by

$$\Phi(\rho) = p_0\rho + p_1 X_4 \rho X_4^\dagger + p_2 Y_4 \rho Y_4^\dagger + p_3 Z_4 \rho Z_4^\dagger,$$

then

$$R^\dagger \Phi \left(R(|00\rangle\langle 00| \otimes \tilde{\rho}) R^\dagger \right) R = |00\rangle\langle 00| \otimes \tilde{\rho} \quad \text{for all} \quad \tilde{\rho} \in M_4,$$

where R is a unitary matrix constructed by the following circuit.



Remark that Φ indeed has a 4-dimensional DFS.

Recent work

Recently, we also considered quantum channels of the form

$$\rho \mapsto \sum_{j=1}^{r} p_j W_j^{\otimes n} \rho W_j^{\otimes n \dagger} \quad \text{where} \quad W_j^{\otimes n} = \underbrace{W_j \otimes \cdots \otimes W_j}_{n}$$

is a tensor product of n copies of unitary matrix $W_j \in M_2$.

Let α, β, γ be any real numbers and let

$$X_\alpha = (e^{i\alpha\sigma_x})^{\otimes 3}, Y_\beta = (e^{i\beta\sigma_y})^{\otimes 3}, Z_\gamma = (e^{i\gamma\sigma_z})^{\otimes 3}.$$

Consider a quantum channel $\Phi : M_8 \to M_8$ given by

$$\Phi(\rho) = p_0\rho + p_1 X_\alpha \rho X_\alpha^\dagger + p_2 Y_\beta \rho Y_\beta^\dagger + p_3 Z_\gamma \rho Z_\gamma^\dagger$$

for some $p_i > 0$ such that $\sum_{i=0}^{3} p_i = 1$.

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Recent work

The 3-qubit case:

Theorem [arXiv:1106.5210]

Let α, β, γ be any real numbers and let

$$X_\alpha = (e^{i\alpha\sigma_x})^{\otimes 3}, Y_\beta = (e^{i\beta\sigma_y})^{\otimes 3}, Z_\gamma = (e^{i\gamma\sigma_z})^{\otimes 3}.$$

Consider a quantum channel $\Phi : M_8 \to M_8$ given by

$$\Phi(\rho) = p_0\rho + p_1 X_\alpha \rho X_\alpha^\dagger + p_2 Y_\beta \rho Y_\beta^\dagger + p_3 Z_\gamma \rho Z_\gamma^\dagger$$

for some $p_i > 0$ such that $\sum_{i=0}^{3} p_i = 1$. Then there is a unitary $U_3 \in M_8$ such that for any data state $\tilde{\rho} \in M_2$,

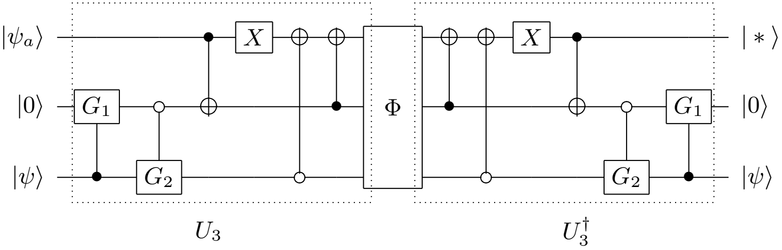
$$\Phi\left(U_3(\rho_a \otimes |0\rangle\langle 0| \otimes \tilde{\rho})U_3^\dagger\right) = U_3\left(\left(\sum_{j=0}^{3} p_j V_j \rho_a V_j^\dagger\right) \otimes |0\rangle\langle 0| \otimes \tilde{\rho}\right)U_3^\dagger, \quad (8)$$

Here ρ_a is an initial single qubit ancilla state and

$$V_0 = I_2,\ V_1 = e^{i\alpha\sigma_x},\ V_2 = e^{i\beta\sigma_y},\ V_3 = e^{i\gamma\sigma_z}.$$

Let U_3 be the 8×8 unitary matrix with columns

$$
\begin{aligned}
|u_1\rangle &= \tfrac{1}{\sqrt{2}}(|100\rangle - |001\rangle) & |u_2\rangle &= \tfrac{1}{\sqrt{6}}(|100\rangle + |001\rangle - 2|010\rangle) \\
|u_3\rangle &= |111\rangle & |u_4\rangle &= \tfrac{1}{\sqrt{3}}(|100\rangle + |001\rangle + |010\rangle) \\
|u_5\rangle &= -(\sigma_x)^{\otimes 3}|u_1\rangle & |u_6\rangle &= -(\sigma_x)^{\otimes 3}|u_2\rangle \\
|u_7\rangle &= -(\sigma_x)^{\otimes 3}|u_3\rangle & |u_8\rangle &= -(\sigma_x)^{\otimes 3}|u_4\rangle
\end{aligned}
$$



$$G_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} \quad \text{and} \quad G_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

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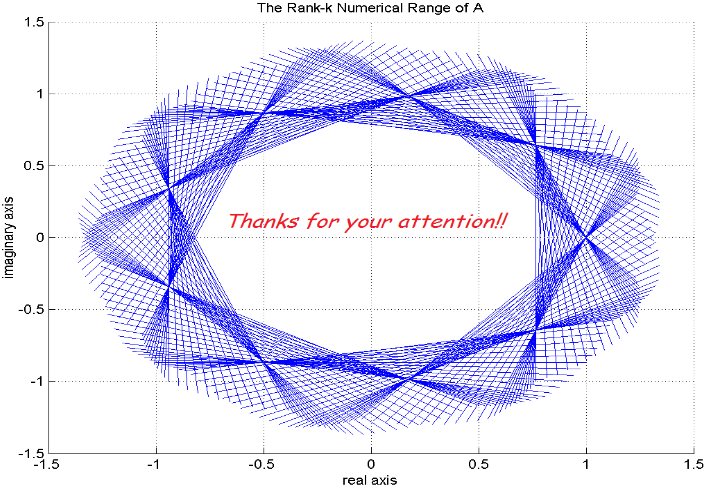
The Rank-k Numerical Range of A
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
imaginary axis
real axis
Thanks for your attention!!