Quantum operations

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2. Completely positive maps on matrix spaces

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3. Cones of positive maps and duality

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4. Decomposable positive maps

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5. Completely positive map and entanglement

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4. Decomposable positive maps
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6. Interpolating problems of completely positive map

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$\Phi_k : M_k(S) \to M_k(\mathcal{B})$, $\Phi_k((A_{ij})) = (\Phi(A_{ij}))$

Φ is k-positive if $\Phi_k(M_k(S)^+) \subseteq M_k(\mathcal{B})^+$

Φ is completely positive if Φ is k-positive for all k.

Dilation and extension of completely positive map

Theorem 1.1

(Stinespring's dilation theorem) Let $\mathcal{A}$ be a unital C^*-algebra, and let $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a linear map.

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Suppose $\mathcal{A}$ is a commutative C^*-algebra and $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is positive. Then Φ is completely positive.

Theorem 1.3

Let S be a self-adjoint subspace of a unital C^*-algebra $\mathcal{A}$ and $\mathcal{B}$ a commutative C^*-algebra. Every positive linear map from S to $\mathcal{B}$ is completely positive.

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Theorem 1.4

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Let $\mathcal{A}$ be a unital C^*-algebra and S be an operator system (norm-closed self-adjoint subspace of $\mathcal{A}$, which contains the identity $1_{\mathcal{A}}$) in $\mathcal{A}$. Then every completely positive map from S to a C^*-algebra $\mathcal{B}$ can be extended to a completely positive map from $\mathcal{A}$ to $\mathcal{B}$.

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Example 1.6 (Arveson [1])

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Example 1.6 (Arveson [1])

Let $C(T)$ be the commutative C^*-algebra of continuous function on the unit circle of the complex plane

Dilation and extension of completely positive map

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Let $\mathcal{A}$ be a unital C^*-algebra and S be an operator system (norm-closed self-adjoint subspace of $\mathcal{A}$, which contains the identity $1_{\mathcal{A}}$) in $\mathcal{A}$. Then every completely positive map from S to a C^*-algebra $\mathcal{B}$ can be extended to a completely positive map from $\mathcal{A}$ to $\mathcal{B}$.

Theorem 1.5

Let $\mathcal{A}$ be a unital C^*-algebra and S an operator system in $\mathcal{A}$. Then every positive map from S to a commutative C^*-algebra $\mathcal{B}$ can be extended to a positive map from $\mathcal{A}$ to $\mathcal{B}$.

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Let $C(T)$ be the commutative C^*-algebra of continuous function on the unit circle of the complex plane and S the subspace of $C(T)$ spanned by $\{1, z, \overline{z}\}$.

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Dilation and extension of completely positive map

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Let $A \in \mathcal{B}(\mathcal{H})$. Then we have

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$$\Psi \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) = \begin{bmatrix} \Phi_{11}(A_{11}) & \Phi_{12}(A_{12}) \\ \Phi_{21}(A_{21}) & \Phi_{22}(A_{22}) \end{bmatrix} .$$

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Therefore, Φ_{12} is an extension of Φ. Since Ψ is completely positive, $\Psi(B)$ is self-adjoint for all self-adjoint $B \in M_2(\mathcal{A})$. In particular, for all $A \in \mathcal{A}$, we have

$$\begin{bmatrix} 0 & \Phi_{12}(A) \\ \Phi_{21}(A^\dagger) & 0 \end{bmatrix} = \Psi\left(\begin{bmatrix} 0 & A \\ A^\dagger & 0 \end{bmatrix}\right)$$

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This shows that Φ_{12} is a contraction.

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This shows that Φ_{12} is a contraction. Similar argument applied to $I_k \otimes \Psi$ shows that $I_k \otimes \Phi_{12}$ is a contraction for all $k \geq 1$.

Therefore, Φ_{12} is an extension of Φ. Since Ψ is completely positive, $\Psi(B)$ is self-adjoint for all self-adjoint $B \in M_2(\mathcal{A})$. In particular, for all $A \in \mathcal{A}$, we have

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This shows that Φ_{12} is a contraction. Similar argument applied to $I_k \otimes \Psi$ shows that $I_k \otimes \Phi_{12}$ is a contraction for all $k \geq 1$. Therefore, Φ_{12} is a complete contractive extension of Φ to $\mathcal{A}$. $\square$

Completely positive maps on matrix spaces

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Let $P_k(n, m)$ denote the set of k-positive map from M_n to M_m. Φ is *completely positive* if $\Phi \in P_k(n, m)$ for all positive integer k.

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To see this, consider $A = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \in M_{2n}^+$. Then

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Example 2.4 (Choi [3])

For every $n > 1$, the map $\Phi : M_n \to M_n$ with

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Let $S_k(n,m) = \{\sum_{i=1}^{k} |x_i\rangle|y_i\rangle : |x_i\rangle \in \mathbf{C}^n,\ |y_i\rangle \in \mathbf{C}^m\}$ be the set of vectors in $\mathbf{C}^n \otimes \mathbf{C}^m$ with Schmidt rank $\le k$.

Completely positive maps on matrix spaces

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On the other hand, in (c), to study $(I_n \otimes P)C(\Phi)(I_n \otimes P) \geq 0$, we only need to consider quadratics in y_j. (see Example 4.2)

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Completely positive maps on matrix spaces

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Completely positive maps on matrix spaces

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Theorem 2.11

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5. $(\mathcal{C}_1 \cap \mathcal{C}_2)^* \supseteq \mathcal{C}_1^* + \mathcal{C}_2^*$, and $(\mathcal{C}_1 \cap \mathcal{C}_2)^* = \mathcal{C}_1^* + \mathcal{C}_2^*$, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed.
6. $(\mathcal{C}_1 + \mathcal{C}_2)^* = \mathcal{C}_1^* \cap \mathcal{C}_2^*$.

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$$\Phi(X) = \sum_{i=1}^r F_i X F_i^\dagger \quad \text{for all } X \in M_n\,.$$

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\hline
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Suppose Φ is decomposable. Then $\Phi = \Theta + \Psi$ where where Θ is completely positive and Ψ is completely copositive. For $k \geq 1$, (A_{ij}) and (A_{ji}) belong to $M_k(\mathcal{A})^+$, $\Theta_k((A_{ij})) \geq 0$ and $\Psi_k((A_{ij})) = (\Psi(A_{ij}))$

Decomposable positive linear maps

Suppose $\Phi : \mathcal{A} \to \mathcal{B}$. For each k, define $\Phi_k^T : M_k \otimes \mathcal{A} \to M_k \otimes \mathcal{B}$ by $\Phi_k^T(C \otimes A) = C^T \otimes \Phi(A)$, and extend by linearity. Note that for $(A_{ij}) \in M_k(A)$, $\Phi_k^T((A_{ij})) = (\Phi(A_{ji})) \in M_k(\mathcal{B}))$. Φ is said to be k-copositive if the map Φ_k^T is positive. Φ is completely copositive if Φ_k^T is positive for all k. Φ is *decomposable* if $\Phi = \Theta + \Psi$, where Θ is completely positive and Ψ is completely copositive. For $n + m \leq 5$, every positive $\Phi : M_n \to M_m$ is decomposable.

Theorem 4.1

Let $\mathcal{A}$ be a C^*-algebra and Φ a linear map of A into $\mathcal{B}(\mathcal{H})$. Then Φ is decomposable if and only if for all $k \geq 1$, whenever (A_{ij}) and (A_{ji}) belong to $M_k(\mathcal{A})^+$ then $(\Phi(A_{ij})) \in M_k(\mathcal{B}(\mathcal{H}))^+$.

Proof of Theorem 4.1

Suppose Φ is decomposable. Then $\Phi = \Theta + \Psi$ where where Θ is completely positive and Ψ is completely copositive. For $k \geq 1$, (A_{ij}) and (A_{ji}) belong to $M_k(\mathcal{A})^+$, $\Theta_k((A_{ij})) \geq 0$ and $\Psi_k((A_{ij})) = (\Psi(A_{ij})) = \Psi_k^T((A_{ji})) \geq 0$.

Decomposable positive linear maps

Suppose $\Phi : \mathcal{A} \to \mathcal{B}$. For each k, define $\Phi_k^T : M_k \otimes \mathcal{A} \to M_k \otimes \mathcal{B}$ by $\Phi_k^T(C \otimes A) = C^T \otimes \Phi(A)$, and extend by linearity. Note that for $(A_{ij}) \in M_k(A)$, $\Phi_k^T((A_{ij})) = (\Phi(A_{ji})) \in M_k(\mathcal{B}))$. Φ is said to be k-copositive if the map Φ_k^T is positive. Φ is completely copositive if Φ_k^T is positive for all k. Φ is *decomposable* if $\Phi = \Theta + \Psi$, where Θ is completely positive and Ψ is completely copositive. For $n + m \leq 5$, every positive $\Phi : M_n \to M_m$ is decomposable.

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Proof of Theorem 4.1

Suppose Φ is decomposable. Then $\Phi = \Theta + \Psi$ where where Θ is completely positive and Ψ is completely copositive. For $k \geq 1$, (A_{ij}) and (A_{ji}) belong to $M_k(\mathcal{A})^+$, $\Theta_k((A_{ij})) \geq 0$ and $\Psi_k((A_{ij})) = (\Psi(A_{ij})) = \Psi_k^T((A_{ji})) \geq 0$. Therefore, $\Phi_k((A_{ij})) \geq 0$.

Decomposable positive linear maps

Decomposable positive linear maps

Proof of Theorem 4.1

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Decomposable positive linear maps

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Decomposable positive linear maps

Proof of Theorem 4.1

Conversely, suppose for all $k \geq 1$, whenever (A_{ij}) and (A_{ji}) belong to $M_k(\mathcal{A})^+$ then $(\Phi(A_{ij})) \in M_k(\mathcal{B}(\mathcal{H}))^+$.
Without loss of generality, we may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$.

Proof of Theorem 4.1

Conversely, suppose for all $k \geq 1$, whenever (A_{ij}) and (A_{ji}) belong to $M_k(\mathcal{A})^+$ then $(\Phi(A_{ij})) \in M_k(\mathcal{B}(\mathcal{H}))^+$.
Without loss of generality, we may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. We can also assume that $I_{\mathcal{B}(\mathcal{K})} \in \mathcal{A}$.

Decomposable positive linear maps

Proof of Theorem 4.1

Conversely, suppose for all $k \geq 1$, whenever (A_{ij}) and (A_{ji}) belong to $M_k(\mathcal{A})^+$ then $(\Phi(A_{ij})) \in M_k(\mathcal{B}(\mathcal{H}))^+$.
Without loss of generality, we may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. We can also assume that $I_{\mathcal{B}(\mathcal{K})} \in \mathcal{A}$. Fix an orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{K}$

Proof of Theorem 4.1

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Without loss of generality, we may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. We can also assume that $I_{\mathcal{B}(\mathcal{K})} \in \mathcal{A}$. Fix an orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{K}$ and let the elements in $\mathcal{B}(\mathcal{K})$ be represented by $A = (a_{ij})$, with $a_{ij} = \langle e_i | A e_j \rangle$.

Proof of Theorem 4.1

Conversely, suppose for all $k \geq 1$, whenever (A_{ij}) and (A_{ji}) belong to $M_k(\mathcal{A})^+$ then $(\Phi(A_{ij})) \in M_k(\mathcal{B}(\mathcal{H}))^+$.
Without loss of generality, we may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. We can also assume that $I_{\mathcal{B}(\mathcal{K})} \in \mathcal{A}$. Fix an orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{K}$ and let the elements in $\mathcal{B}(\mathcal{K})$ be represented by $A = (a_{ij})$, with $a_{ij} = \langle e_i|Ae_j\rangle$. Then we can define the transpose in $\mathcal{B}(\mathcal{K})$, $A^T = (a_{ji})$.

Decomposable positive linear maps

Proof of Theorem 4.1

Conversely, suppose for all $k \geq 1$, whenever (A_{ij}) and (A_{ji}) belong to $M_k(\mathcal{A})^+$ then $(\Phi(A_{ij})) \in M_k(\mathcal{B}(\mathcal{H}))^+$.
Without loss of generality, we may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. We can also assume that $I_{\mathcal{B}(\mathcal{K})} \in \mathcal{A}$. Fix an orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{K}$ and let the elements in $\mathcal{B}(\mathcal{K})$ be represented by $A = (a_{ij})$, with $a_{ij} = \langle e_i|Ae_j\rangle$. Then we can define the transpose in $\mathcal{B}(\mathcal{K})$, $A^T = (a_{ji})$. Let

$$S = \{\begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix} \in M_2(\mathcal{B}(\mathcal{K})) : A \in \mathcal{A}\}.$$

Proof of Theorem 4.1

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Without loss of generality, we may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. We can also assume that $I_{\mathcal{B}(\mathcal{K})} \in \mathcal{A}$. Fix an orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{K}$ and let the elements in $\mathcal{B}(\mathcal{K})$ be represented by $A = (a_{ij})$, with $a_{ij} = \langle e_i | A e_j \rangle$. Then we can define the transpose in $\mathcal{B}(\mathcal{K})$, $A^T = (a_{ji})$. Let $S = \left\{ \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix} \in M_2(\mathcal{B}(\mathcal{K})) : A \in \mathcal{A} \right\}$. Then S is an operator system in $M_2(\mathcal{B}(\mathcal{K}))$.

Decomposable positive linear maps

Proof of Theorem 4.1

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Without loss of generality, we may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. We can also assume that $I_{\mathcal{B}(\mathcal{K})} \in \mathcal{A}$. Fix an orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{K}$ and let the elements in $\mathcal{B}(\mathcal{K})$ be represented by $A = (a_{ij})$, with $a_{ij} = \langle e_i | A e_j \rangle$. Then we can define the transpose in $\mathcal{B}(\mathcal{K})$, $A^T = (a_{ji})$. Let $S = \{ \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix} \in M_2(\mathcal{B}(\mathcal{K})) : A \in \mathcal{A}\}$. Then S is an operator system in $M_2(\mathcal{B}(\mathcal{K}))$. Define $\Psi : S \to \mathcal{B}(\mathcal{H})\}$ by

$$\Psi\left(\begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}\right) = \Phi(A).$$

Decomposable positive linear maps

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Proof of Theorem 4.1

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Decomposable positive linear maps

Proof of Theorem 4.1

For each $k \geq 1$, suppose $\left(A_{ij} \oplus A_{ij}^T\right) \in M_k(S)$ is positive. Then $(A_{ij})\,,\ \left(A_{ij}^T\right) \geq 0$. Since

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$$\left(A_{ij}^T\right) \geq 0 \Rightarrow\ (A_{ji}) = \left(A_{ij}^T\right)^T \geq 0,$$

we have $\left(\Psi\left(A_{ij} \oplus A_{ij}^T\right)\right) = (\Phi(A_{ij})) \geq 0.$

Decomposable positive linear maps

Proof of Theorem 4.1

For each $k \geq 1$, suppose $\left(A_{ij} \oplus A_{ij}^T\right) \in M_k(S)$ is positive. Then (A_{ij}), $\left(A_{ij}^T\right) \geq 0$. Since

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we have $\left(\Psi\left(A_{ij} \oplus A_{ij}^T\right)\right) = (\Phi(A_{ij})) \geq 0$. So, Ψ is k-positive.

Decomposable positive linear maps

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Decomposable positive linear maps

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Decomposable positive linear maps

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$$\Theta_1(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

Proof of Theorem 4.1

For each $k \geq 1$, suppose $\left(A_{ij} \oplus A_{ij}^T\right) \in M_k(S)$ is positive. Then $\left(A_{ij}\right), \left(A_{ij}^T\right) \geq 0$. Since

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we have $\left(\Psi\left(A_{ij} \oplus A_{ij}^T\right)\right) = \left(\Phi(A_{ij})\right) \geq 0$. So, Ψ is k-positive. Hence, Ψ is completely positive on the operator system S. By Theorem 1.4, Ψ can be extended to a completely positive map on $M_2(\mathcal{B}(\mathcal{H}))$. Let $\Theta_1, \Theta_2 : \mathcal{A} \to M_2(\mathcal{B}(\mathcal{K}))$ be given by

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Then Θ_1 is completely positive

Proof of Theorem 4.1

For each $k \geq 1$, suppose $\left(A_{ij} \oplus A_{ij}^T\right) \in M_k(S)$ is positive. Then $\left(A_{ij}\right), \ \left(A_{ij}^T\right) \geq 0$. Since

$$\left(A_{ij}^T\right) \geq 0 \Rightarrow \ (A_{ji}) = \left(A_{ij}^T\right)^T \geq 0,$$

we have $\left(\Psi\left(A_{ij} \oplus A_{ij}^T\right)\right) = (\Phi(A_{ij})) \geq 0$. So, Ψ is k-positive. Hence, Ψ is completely positive on the operator system S. By Theorem 1.4, Ψ can be extended to a completely positive map on $M_2(\mathcal{B}(\mathcal{H}))$. Let $\Theta_1, \ \Theta_2 : \mathcal{A} \to M_2(\mathcal{B}(\mathcal{K}))$ be given by

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Then Θ_1 is completely positive and Θ_2 is completely copositive.

Proof of Theorem 4.1

For each $k \geq 1$, suppose $\left(A_{ij} \oplus A_{ij}^T\right) \in M_k(S)$ is positive. Then (A_{ij}), $\left(A_{ij}^T\right) \geq 0$. Since

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we have $\left(\Psi\left(A_{ij} \oplus A_{ij}^T\right)\right) = (\Phi(A_{ij})) \geq 0$. So, Ψ is k-positive. Hence, Ψ is completely positive on the operator system S. By Theorem 1.4, Ψ can be extended to a completely positive map on $M_2(\mathcal{B}(\mathcal{H}))$. Let $\Theta_1,\ \Theta_2 : \mathcal{A} \to M_2(\mathcal{B}(\mathcal{K}))$ be given by

$$\Theta_1(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } \Theta_2(A) = \begin{bmatrix} 0 & 0 \\ 0 & A^T \end{bmatrix}.$$

Then Θ_1 is completely positive and Θ_2 is completely copositive. Therefore, $\Phi = \Psi \circ (\Theta_1 + \Theta_2)$ is decomposable.

Decomposable positive linear maps

Decomposable positive linear maps

Example 4.2 Choi [5]

Decomposable positive linear maps

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Let $\Phi : M_3 \to M_3$ be given by

$$\Phi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$$

Decomposable positive linear maps

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$$\Phi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$$

$$= \begin{bmatrix} a_{11} + 2a_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} + 2a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} + 2a_{22} \end{bmatrix}.$$

Decomposable positive linear maps

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Then Φ is positive but not indecomposable.

Decomposable positive linear maps

Proof of Example 4.2

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To prove that Φ is positive, we use Theorem 2.5 (a) $\Leftrightarrow$ (c) for $k = 1$.

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$$C(\Phi) = \left[\begin{array}{rrr|rrr|rrr}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{array}\right]$$

Decomposable positive linear maps

Decomposable positive linear maps

Proof of Example 4.2

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Proof of Example 4.2

Suppose $P = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \overline{x_1} & \overline{x_2} & \overline{x_3} \end{bmatrix}$ is a rank one orthogonal projection.

Proof of Example 4.2

Suppose $P = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \overline{x_1} & \overline{x_2} & \overline{x_3} \end{bmatrix}$ is a rank one orthogonal projection.

Then

$$(I_3 \otimes P)C(\Phi)(I_3 \otimes P) = X \otimes P,$$

Decomposable positive linear maps

Proof of Example 4.2

Suppose $P = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \overline{x_1} & \overline{x_2} & \overline{x_3} \end{bmatrix}$ is a rank one orthogonal projection. Then

$$(I_3 \otimes P)C(\Phi)(I_3 \otimes P) = X \otimes P, \text{ where}$$

$$X = \begin{bmatrix} |x_1|^2 + 2|x_2|^2 & -\overline{x_1}x_2 & -\overline{x_1}x_3 \\ -x_1\overline{x_2} & |x_2|^2 + 2|x_3|^2 & -\overline{x_2}x_3 \\ -x_1\overline{x_3} & -x_2\overline{x_3} & |x_3|^2 + 2|x_1|^2 \end{bmatrix}.$$

Decomposable positive linear maps

Decomposable positive linear maps

Proof of Example 4.2

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Since

$$|x_1|^2 + 2|x_2|^2 \geq 0,$$

Decomposable positive linear maps

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$$\det\left(\begin{bmatrix} |x_1|^2 + 2|x_2|^2 & -\overline{x_1}x_2 \\ -x_1\overline{x_2} & |x_2|^2 + 2|x_3|^2 \end{bmatrix}\right)$$

$$= 2(|x_2|^4 + |x_3|^3(|x_1|^2 + 2|x_2|^2)) \geq 0$$

Proof of Example 4.2

Since

$$|x_1|^2 + 2|x_2|^2 \geq 0,$$

$$\det\left(\begin{bmatrix} |x_1|^2 + 2|x_2|^2 & -\overline{x_1}x_2 \\ -x_1\overline{x_2} & |x_2|^2 + 2|x_3|^2 \end{bmatrix}\right)$$

$$= 2(|x_2|^4 + |x_3|^3(|x_1|^2 + 2|x_2|^2)) \geq 0$$

$$\det(X) = 4\left(|x_1|^2|x_2|^4 + |x_1|^4|x_3|^2 + |x_1|^2|x_2|^2|x_3|^2 + |x_2|^2|x_3|^4\right) \geq 0\,,$$

Decomposable positive linear maps

Proof of Example 4.2

Since

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$$= 2(|x_2|^4 + |x_3|^3(|x_1|^2 + 2|x_2|^2)) \geq 0$$

$$\det(X) = 4\left(|x_1|^2|x_2|^4 + |x_1|^4|x_3|^2 + |x_1|^2|x_2|^2|x_3|^2 + |x_2|^2|x_3|^4\right) \geq 0\,,$$

we have $X \geq 0$.

Proof of Example 4.2

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$$= 2(|x_2|^4 + |x_3|^3(|x_1|^2 + 2|x_2|^2)) \geq 0$$

$$\det(X) = 4\left(|x_1|^2|x_2|^4 + |x_1|^4|x_3|^2 + |x_1|^2|x_2|^2|x_3|^2 + |x_2|^2|x_3|^4\right) \geq 0\,,$$

we have $X \geq 0$. Hence, $(I_3 \otimes P)C(\Phi)(I_3 \otimes P) = X \otimes P \geq 0$.

Proof of Example 4.2

Since

$$|x_1|^2 + 2|x_2|^2 \geq 0,$$

$$\det\left(\begin{bmatrix} |x_1|^2 + 2|x_2|^2 & -\overline{x_1}x_2 \\ -x_1\overline{x_2} & |x_2|^2 + 2|x_3|^2 \end{bmatrix}\right)$$

$$= 2(|x_2|^4 + |x_3|^3(|x_1|^2 + 2|x_2|^2)) \geq 0$$

$$\det(X) = 4\left(|x_1|^2|x_2|^4 + |x_1|^4|x_3|^2 + |x_1|^2|x_2|^2|x_3|^2 + |x_2|^2|x_3|^4\right) \geq 0\,,$$

we have $X \geq 0$. Hence, $(I_3 \otimes P)C(\Phi)(I_3 \otimes P) = X \otimes P \geq 0$. By Theorem 2.5, Φ is positive.

Proof of Example 4.2

Decomposable positive linear maps

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Next, we will use Theorem 4.1 to show that Φ is not decomposable.

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Next, we will use Theorem 4.1 to show that Φ is not decomposable.
Let $(x_{ij}) \in M_3(M_3)$ be given by

$$(x_{ij}) = \left[\begin{array}{ccc|ccc|ccc} 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \end{array}\right]$$

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It is easy to check that (x_{ij}) and (x_{ji}) are positive but

Decomposable positive linear maps

Proof of Example 4.2

Decomposable positive linear maps

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$$\Phi((x_{ij})) = \left[\begin{array}{ccc|ccc|ccc} \mathbf{6} & 0 & 0 & 0 & -\mathbf{4} & 0 & 0 & 0 & -\mathbf{4} \\ 0 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 33 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 33 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{4} & 0 & 0 & 0 & \mathbf{6} & 0 & 0 & 0 & -\mathbf{4} \\ 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 33 & 0 \\ -\mathbf{4} & 0 & 0 & 0 & -\mathbf{4} & 0 & 0 & 0 & \mathbf{6} \end{array}\right]$$

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is not positive because -2 is an eigenvalue of $\Phi((x_{ij}))$. $\square$

Completely positive maps and entanglement

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Lemma 5.3

A state $\rho \in M_n \otimes M_m$ is separable if and only if $\mathrm{Tr}(\rho A) \geq 0$ for all $A \in M_{mn}$ such that $\mathrm{Tr}(A(P \otimes Q)) \geq 0$ for all orthogonal projections $P \in M_n$ and $Q \in M_m$.

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$$E = \sum_{i,j=1}^{n} E_{i\,j} \otimes E_{i\,j} = \left(\sum_{i=1} |e_i\rangle|e_i\rangle\right)\left(\sum_{j=1} |e_j\rangle|e_j\rangle\right)^\dagger$$

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$$C(\Phi) = (I_n \otimes \Phi)(E)\,.$$

Completely positive maps and entanglement

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Hence,

$$
\begin{aligned}
& (I_n \otimes \Phi)(\rho) \geq 0 \\
\Rightarrow \quad & \langle E | (I_n \otimes \Phi)(\rho) \rangle \geq 0 \\
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So, by lemma 5.3, ρ is separable.

Completely positive maps and entanglement

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Define two partial transpose map on $M_n \otimes M_m$ by

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(1) If ρ is separable, then $T_2(\rho) \geq 0$.

(2) If $n + m \leq 5$ and $T_2(\rho) \geq 0$, then ρ is separable.

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Completely positive maps and entanglement

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Then $(I \otimes \Phi_1)(\rho) \geq 0$ and $(I \otimes \Phi_2)(\rho) = (I \otimes \Phi_2^T)(T_2(\rho)) \geq 0$.

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To proof (2), suppose $n + m \leq 5$ and $T_2(\rho) \geq 0$. Let $\Phi : M_m \to M_n$ be a positive map. Then $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1 : M_m \to M_n$ is completely positive and $\Phi_2 : M_m \to M_n$ is completely copositive.

Then $(I \otimes \Phi_1)(\rho) \geq 0$ and $(I \otimes \Phi_2)(\rho) = (I \otimes \Phi_2^T)(T_2(\rho)) \geq 0$.

Hence, $(I \otimes \Phi)(\rho) \geq 0$.

Proof of Theorem 5.4

Note that $T_1(\rho) = (T_2(\rho))^T$. Therefore, the condition $T_2(\rho) \geq 0$ is equivalent to $T_1(\rho) \geq 0$.

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Hence, $(I \otimes \Phi)(\rho) \geq 0$. So, by Theorem 5.1, Φ is completely positive. $\square$

To show that the conclusion in Theorem 5,4 (b) may not hold for $n = m = 3$,

Completely positive maps and entanglement

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$$\rho = \frac{1}{63}\left[\begin{array}{ccc|ccc|ccc} 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \end{array}\right]$$

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Then by the discussion in Example 4.2, ρ, $T_2(\rho) \geq 0$.

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Then by the discussion in Example 4.2, ρ, $T_2(\rho) \geq 0$. So, ρ is a PPT state but $(I \otimes \Phi)(\rho) \not\geq 0$. Therefore, by Theorem 5.1, ρ is not separable.

An Interpolating Problem

An Interpolating Problem

Problem 6.1

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Given $A = (a_{ij}) \in M_n$, let $\mathrm{vec}\,(A) = (a_{11}, \ldots, a_{1n}, \ldots, a_{21}, \ldots, a_{nn}) \in \mathbf{C}^{n^2}$.

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$A \to \mathrm{vec}\,(A)$ gives a linear isomorphism between M_n and $\mathbf{C}^{n^2}$. Let $C = (C_{ij}) \in M_n(M_m)$, the realignment matrix of C is given by

$$C^R = \begin{bmatrix} \mathrm{vec}\,(C_{11}) \\ \mathrm{vec}\,(C_{12}) \\ \vdots \\ \mathrm{vec}\,(C_{1n}) \\ \mathrm{vec}\,(C_{21}) \\ \vdots \\ \mathrm{vec}\,(C_{nn}). \end{bmatrix}$$

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An Interpolating Problem

An Interpolating Problem

Theorem 6.2

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An Interpolating Problem

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A *nonnegative* matrix is column (respectively, row)

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A *nonnegative* matrix is column (respectively, row) stochastic

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A *nonnegative* matrix is column (respectively, row) stochastic if in each column (respectively, row) the entries sum up to 1.

An Interpolating Problem

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A *nonnegative* matrix is column (respectively, row) stochastic if in each column (respectively, row) the entries sum up to 1. If A is both column and row stochastic, then (the necessarily square matrix) A is doubly stochastic.

An Interpolating Problem

Theorem 6.2

Suppose $A_i,\ B_i$ are diagonal matrices with diagonals $\mathbf{a}_i,\ \mathbf{b}_i$. Then the following conditions are equivalent:

1. There exists a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A_i) = B_i$ for all $1 \leq i \leq k$.
2. There exists an $n \times m$ nonnegative matrix D such that $\mathbf{b}_i = \mathbf{a}_i D$ for all $1 \leq i \leq k$.

A *nonnegative* matrix is column (respectively, row) stochastic if in each column (respectively, row) the entries sum up to 1. If A is both column and row stochastic, then (the necessarily square matrix) A is doubly stochastic.

Theorem 6.3

An Interpolating Problem

Theorem 6.2

Suppose $A_i,\ B_i$ are diagonal matrices with diagonals $\mathbf{a}_i,\ \mathbf{b}_i$. Then the following conditions are equivalent:

1. There exists a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A_i) = B_i$ for all $1 \leq i \leq k$.
2. There exists an $n \times m$ nonnegative matrix D such that $\mathbf{b}_i = \mathbf{a}_i D$ for all $1 \leq i \leq k$.

A *nonnegative* matrix is column (respectively, row) stochastic if in each column (respectively, row) the entries sum up to 1. If A is both column and row stochastic, then (the necessarily square matrix) A is doubly stochastic.

Theorem 6.3

Φ in Theorem 6.2 can be choose to be unital

An Interpolating Problem

Theorem 6.2

Suppose A_i, B_i are diagonal matrices with diagonals $\mathbf{a}_i$, $\mathbf{b}_i$. Then the following conditions are equivalent:

1. There exists a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A_i) = B_i$ for all $1 \leq i \leq k$.
2. There exists an $n \times m$ nonnegative matrix D such that $\mathbf{b}_i = \mathbf{a}_i D$ for all $1 \leq i \leq k$.

A *nonnegative* matrix is column (respectively, row) stochastic if in each column (respectively, row) the entries sum up to 1. If A is both column and row stochastic, then (the necessarily square matrix) A is doubly stochastic.

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An Interpolating Problem

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(c) There are real numbers $\gamma_1, \gamma_2 \geq 0$ such that

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Let $A \in H_n$ and $B \in H_m$. Then the following conditions are equivalent.

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Let $A \in H_n$ and $B \in H_m$. Then the following conditions are equivalent.

(a) There is a completely positive linear map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$.

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Example 6.5

Let $A = \operatorname{diag}(2, 1, 0)$, $B_1 = \operatorname{diag}(4, 3, 1)$ and $B_2 = \operatorname{diag}(1, 1, -1)$. There is a completely positive linear map Φ such that $\Phi(A) = B_1$, but there is no completely positive linear map Φ such that $\Phi(A) = B_2$.

An Interpolating Problem

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Theorem 6.6

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Theorem 6.7

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Suppose $A \in H_n$ and $B \in H_m$. Denote by $\lambda_+(X)$ the sum of positive eigenvalues of a Hermitian matrix X.

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Theorem 6.7

Suppose $A \in H_n$ and $B \in H_m$. Denote by $\lambda_+(X)$ the sum of positive eigenvalues of a Hermitian matrix X. The following conditions are equivalent.

(a) There is a trace preserving completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$.

An Interpolating Problem

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Let $A \in H_n$ and $B \in H_m$. The following conditions are equivalent.

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(b) There exists an $n \times m$ column stochastic matrix D such that $\lambda(B) = \lambda(A)D$.

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An Interpolating Problem

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(a) There exists a unital completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$.

(b) There exists an $n \times m$ column stochastic matrix D such that $\lambda(B) = \lambda(A)D$.

(c) $\lambda_n(A) \leq \lambda_i(B) \leq \lambda_1(A)$ for all $1 \leq i \leq m$.

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Suppose $A \in H_n$ and $B \in H_m$. Denote by $\lambda_+(X)$ the sum of positive eigenvalues of a Hermitian matrix X. The following conditions are equivalent.

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(b) There exists an $n \times m$ row stochastic matrix D such that $\lambda(B) = \lambda(A)D$.

(c) $\lambda_+(B) \leq \lambda_+(A)$, and $\mathrm{Tr} A = \mathrm{Tr} B$.

An Interpolating Problem

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Example 6.8

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Let $A = \operatorname{diag}(2, 1, -1)$, $B = \operatorname{diag}(2, 0, 0)$, and $C = \operatorname{diag}(1, 1, 0)$.

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Let $A = \operatorname{diag}(2, 1, -1)$, $B = \operatorname{diag}(2, 0, 0)$, and $C = \operatorname{diag}(1, 1, 0)$. Then there are trace preserving completely positive linear maps Φ_1, Φ_2 such that $\Phi_1(A) = B$, $\Phi_2(B) = C$,

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Example 6.8

Let $A = \mathrm{diag}\,(2, 1, -1)$, $B = \mathrm{diag}\,(2, 0, 0)$, and $C = \mathrm{diag}\,(1, 1, 0)$. Then there are trace preserving completely positive linear maps Φ_1, Φ_2 such that $\Phi_1(A) = B$, $\Phi_2(B) = C$, and $\Phi_2 \circ \Phi_1(A) = C$. There is no completely positive linear map Φ satisfying $\Phi(C) = A$.

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Let $A = \operatorname{diag}(2, 1, -1)$, $B = \operatorname{diag}(2, 0, 0)$, and $C = \operatorname{diag}(1, 1, 0)$. Then there are trace preserving completely positive linear maps Φ_1, Φ_2 such that $\Phi_1(A) = B$, $\Phi_2(B) = C$, and $\Phi_2 \circ \Phi_1(A) = C$. There is no completely positive linear map Φ satisfying $\Phi(C) = A$.

Remark 6.9

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Let $A = \text{diag}\,(2, 1, -1)$, $B = \text{diag}\,(2, 0, 0)$, and $C = \text{diag}\,(1, 1, 0)$. Then there are trace preserving completely positive linear maps Φ_1, Φ_2 such that $\Phi_1(A) = B$, $\Phi_2(B) = C$, and $\Phi_2 \circ \Phi_1(A) = C$. There is no completely positive linear map Φ satisfying $\Phi(C) = A$.

Remark 6.9

For two density matrices A and B,

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Let $A = \mathrm{diag}\,(2, 1, -1)$, $B = \mathrm{diag}\,(2, 0, 0)$, and $C = \mathrm{diag}\,(1, 1, 0)$. Then there are trace preserving completely positive linear maps Φ_1, Φ_2 such that $\Phi_1(A) = B$, $\Phi_2(B) = C$, and $\Phi_2 \circ \Phi_1(A) = C$. There is no completely positive linear map Φ satisfying $\Phi(C) = A$.

Remark 6.9

For two density matrices A and B, there is always a trace preserving completely positive map such that $\Phi(A) = B$.

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Example 6.8

Let $A = \operatorname{diag}(2,1,-1)$, $B = \operatorname{diag}(2,0,0)$, and $C = \operatorname{diag}(1,1,0)$. Then there are trace preserving completely positive linear maps Φ_1, Φ_2 such that $\Phi_1(A) = B$, $\Phi_2(B) = C$, and $\Phi_2 \circ \Phi_1(A) = C$. There is no completely positive linear map Φ satisfying $\Phi(C) = A$.

Remark 6.9

For two density matrices A and B, there is always a trace preserving completely positive map such that $\Phi(A) = B$. But there may not be a unital completely positive map Ψ such that $\Psi(A) = B$.

An Interpolating Problem

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Example 6.10

Suppose there is a unital completely positive map taking A to B, and also a trace preserving completely positive map taking A to B. Is there a unital trace preserving completely positive map sending A to B? The following example shows that the answer is negative.

Example 6.10

Suppose $A = \mathrm{diag}\,(4,1,1,0)$ and $B = \mathrm{diag}\,(3,3,0,0)$.

An Interpolating Problem

Suppose there is a unital completely positive map taking A to B, and also a trace preserving completely positive map taking A to B. Is there a unital trace preserving completely positive map sending A to B? The following example shows that the answer is negative.

Example 6.10

Suppose $A = \operatorname{diag}(4,1,1,0)$ and $B = \operatorname{diag}(3,3,0,0)$. By Theorems 6.6 and 6.7 there is a trace preserving completely positive map sending A to B,

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Suppose $A = \operatorname{diag}(4, 1, 1, 0)$ and $B = \operatorname{diag}(3, 3, 0, 0)$. By Theorems 6.6 and 6.7 there is a trace preserving completely positive map sending A to B, and also a unital completely positive map sending A to B.

An Interpolating Problem

Suppose there is a unital completely positive map taking A to B, and also a trace preserving completely positive map taking A to B. Is there a unital trace preserving completely positive map sending A to B? The following example shows that the answer is negative.

Example 6.10

Suppose $A = \operatorname{diag}(4,1,1,0)$ and $B = \operatorname{diag}(3,3,0,0)$. By Theorems 6.6 and 6.7 there is a trace preserving completely positive map sending A to B, and also a unital completely positive map sending A to B. Let
$A_1 = A - I_4 = \operatorname{diag}(3,0,0,-1)$ and $B_1 = B - I_4 = \operatorname{diag}(2,2,-1,-1)$.

An Interpolating Problem

Suppose there is a unital completely positive map taking A to B, and also a trace preserving completely positive map taking A to B. Is there a unital trace preserving completely positive map sending A to B? The following example shows that the answer is negative.

Example 6.10

Suppose $A = \operatorname{diag}(4, 1, 1, 0)$ and $B = \operatorname{diag}(3, 3, 0, 0)$. By Theorems 6.6 and 6.7 there is a trace preserving completely positive map sending A to B, and also a unital completely positive map sending A to B. Let $A_1 = A - I_4 = \operatorname{diag}(3, 0, 0, -1)$ and $B_1 = B - I_4 = \operatorname{diag}(2, 2, -1, -1)$. By Theorem 6.7, there is no trace preserving completely positive linear map sending A_1 to B_1.

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Suppose there is a unital completely positive map taking A to B, and also a trace preserving completely positive map taking A to B. Is there a unital trace preserving completely positive map sending A to B? The following example shows that the answer is negative.

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Suppose $A = \mathrm{diag}\,(4, 1, 1, 0)$ and $B = \mathrm{diag}\,(3, 3, 0, 0)$. By Theorems 6.6 and 6.7 there is a trace preserving completely positive map sending A to B, and also a unital completely positive map sending A to B. Let $A_1 = A - I_4 = \mathrm{diag}\,(3, 0, 0, -1)$ and $B_1 = B - I_4 = \mathrm{diag}\,(2, 2, -1, -1)$. By Theorem 6.7, there is no trace preserving completely positive linear map sending A_1 to B_1. Hence, there is no unital trace preserving completely positive map sending A to B.

An Interpolating Problem

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A quantum channel/completely positive map $\Phi : M_n \to M_n$ is called mixed unitary (mixing process) if there exist unitary matrices $U_1, \dots, U_r \in M_n$ and positive numbers $p_1, \dots, p_r$ summing up to 1

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An Interpolating Problem

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Example 6.11

$(3, 2, 1, 0) \prec (6, 1, 0, -1)$, $(3, 3, 0, 0) \not\prec (4, 1, 1, 0)$.

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Let $A, B \in H_n$. The following are equivalent.

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(f) There is a doubly stochastic matrix D such that $\lambda(B) = \lambda(A)D$.

Thank you

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