Summer School on Quantum Information Science
Taiyuan University of Technology

Yiu Tung Poon
Department of Mathematics, Iowa State University,
Ames, Iowa 50011, USA
(ytpoon@iastate.edu).

# Quantum operations

# Table of content

# 1 Dilation and extension of completely positive map

Mathematically, quantum operations (channels) are completely positive linear maps. The basic references for completely positive map and its connection with quantum information are [3], [4], [13] [14] and [15]. For the consistence with other lectures, we will use $|x\rangle$ for (column) vectors and $A^\dagger$ for the adjoint of an operator $A$ in the following discussion.

In this section, we study dilation and extension of completely positive maps on $C^*$-algebras. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$, the set of bounded linear operators on $\mathcal{H}$. A $C^*$-algebra $\mathcal{A}$ is a complex Banach algebra with a conjugate linear map $A \to A^\dagger$ satisfying $\|AA^\dagger\| = \|A\|^2$ for all $A \in \mathcal{A}$. By the GNS-construction [8], every $C^*$-algebra is isomorphic to a norm closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$. Every commutative $C^*$-algebra is isomorphic to some $C_0(X)$, the algebra of continuous function vanishing at infinity on a locally compact Haudorff space $X$.

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $\langle x|Ax\rangle \geq 0$ for all $|x\rangle \in \mathcal{H}$. The set of positive operators in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{B}(\mathcal{H})^+$. For each $k \geq 1$, we can identify $M_k(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^k)$, the bounded linear operators on $\mathcal{H}^k$ and the positive operators in $M_k(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^k)^+$. By identifying a $C^*$-algebra $\mathcal{A}$ as a norm closed self-adjoint subalgebra of some $\mathcal{B}(\mathcal{H})$, we can defined the positive elements of $M_k(\mathcal{A})$. A linear map $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is $k$-positive if $I_k \otimes \Phi$ maps the positive elements of $M_k(\mathcal{A})$ to the positive elements of $M_n(\mathcal{B}(\mathcal{H}))$. $\Phi$ is *completely positive* if $\Phi$ is $k$-positive for all $k$. Following is the most fundamental result on completely positive maps [16].

**Theorem 1.1** *(Stinespring's dilation theorem) Let $\mathcal{A}$ be a unital $C^*$-algebra, and let $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a linear map. Then $\Phi$ is completely positive if and only if there exist a Hilbert space $\mathcal{K}$, a unital $C^*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, and a bounded operator $V : H \to \mathcal{K}$ such that*

$$\Phi(A) = V^\dagger \pi(A) V \quad \textit{for all } A \in \mathcal{A}.$$

We will outline the proof of this theorem in the exercise. For commutative $A$, we have the following [17].

**Theorem 1.2** *Suppose $\mathcal{A}$ is a commutative $C^*$-algebra and $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is positive. Then $\Phi$ is completely positive.*

One of the application of the study of completely positive maps is the extension problem for positive maps on $C^*$-algebra. Let $\mathcal{A}$ be a unital $C^*$-algebra and $S$ a self-adjoint ($S = S^\dagger$) subspace of $\mathcal{A}$. A linear map $\Phi : S \to \mathcal{B}(\mathcal{H})$ is said to be positive if $\Phi(A)$ is positive in $\mathcal{B}(\mathcal{H})$ for every positive $A \in S$. The extension problem asks if $\Phi$ can be extended to a positive map on $\mathcal{A}$. When the range of $\Phi$ is commutative, the answer is affirmative. We first state some results of Arveson [1].

**Theorem 1.3** *Let $S$ be a self-adjoint subspace of a $C^*$-algebra $\mathcal{A}$ and $\mathcal{B}$ a commutative $C^*$-algebra. Every positive linear map from $S$ to $\mathcal{B}$ is completely positive.*

**Theorem 1.4** *Let $\mathcal{A}$ be a unital $C^*$-algebra and $S$ be a norm-closed self-adjoint subspace of $\mathcal{A}$, which contains the identity $1_{\mathcal{A}}$ in $\mathcal{A}$. Then every completely positive map from $S$ to a $C^*$-algebra $\mathcal{B}$ can be extended to a completely positive map from $\mathcal{A}$ to $\mathcal{B}$.*

**Theorem 1.5** *Let $\mathcal{A}$ be a unital $C^*$-algebra and $S$ be a norm-closed self-adjoint subspace of $\mathcal{A}$, which contains the identity in $\mathcal{A}$. Then every positive map from $S$ to a commutative $C^*$-algebra $\mathcal{B}$ can be extended to a positive map from $\mathcal{A}$ to $\mathcal{B}$.*

The following example shows that Theorem 1.4 does not hold for positive maps. It also shows that Theorem 1.3 does not hold if we exchange the commutativity of $\mathcal{B}$ with $\mathcal{A}$. See also Theorem 1.2.

**Example 1.6** (Arveson [1]) Let $C(T)$ be the commutative $C^*$-algebra of continuous function on the unit circle of the complex plane and $S$ the subspace of $C(T)$ spanned by $\{1, z, \overline{z}\}$. Define $\Phi : S \to M_2$ by

$$\Phi(a + bz + c\overline{z}) = \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}.$$

Then $\Phi$ is positive (exercise). We are going to show that $\Phi$ cannot be extended to a positive map on $C(T)$.

Suppose the contrary that $\Phi$ can be extended to a positive map on $C(T)$. Then by Theorem 1.2, $\Phi : C(T) \to M_2$ is completely positive. By Theorem 1.1, there exist a Hilbert space $\mathcal{K}$, a unital $C^*$-homomorphism $\pi : C(T) \to \mathcal{B}(\mathcal{K})$, and a bounded operator $V : \mathbf{C}^2 \to \mathcal{K}$ such that

$$\Phi(A) = V^\dagger \pi(A) V \quad \text{for all } A \in C(T).$$

Since $\Phi$ is unital, $V^\dagger V = I_2$. Therefore, $VV^\dagger$ is a projection in $\mathcal{B}(\mathcal{K})$. So, we have

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \Phi(1) = \Phi(z\overline{z}) = V^\dagger \pi(z\overline{z}) V = V^\dagger \pi(z)\pi(\overline{z}) V \\ &\geq \; V^\dagger \pi(z) V V^\dagger \pi(\overline{z}) V = \Phi(z)\Phi(\overline{z}) \\ &= \; \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

a contradiction. □

We remark that in Theorem 1.4, the condition $1_{\mathcal{A}} \in S$ is also important because it ensures that every self adjoint element in $S$ is a difference of two positive elements. Without this condition, the Theorem would not hold (exercise).

A norm closed subspace $S$ of a $C^*$-algebra $\mathcal{A}$ is called an *operator space* of $A$. A linear map $\Phi : S \to \mathcal{B}(\mathcal{H})$ is called a *complete contraction* if $\Phi_k$ is a contraction ($\|\Phi_k\| \leq 1$) for every $k$. If an operator space $S$ is self-adjoint and $1_{\mathcal{A}} \in S$, then $S$ is called an *operator system.* There is a close connection between the study of complete contraction on operator system and completely positive map on operator system. This stems from the connection between the norm and positivity in $\mathcal{B}(\mathcal{H})$.

**Lemma 1.7** *(Choi and Effros, [6]) Let $A \in \mathcal{B}(\mathcal{H})$. Then we have*

$$\|A\| \leq 1 \Leftrightarrow \begin{bmatrix} I & A \\ A^\dagger & I \end{bmatrix} \geq 0\,.$$

We leave the proof as an exercise.

The following theorem [2, 15] is an analog to Theorem 1.4.

**Theorem 1.8** *Let $S$ be an operator space in a $C^*$-algebra $\mathcal{A}$ and $\Phi : S \to \mathcal{B}(\mathcal{H})$ a complete contraction. Then $\Phi$ can be extended to a complete contraction on $\mathcal{A}$*

*Proof.* Without loss of generality, we may assume that $\mathcal{A}$ is unital. Let

$$S_2 = \{ \begin{bmatrix} \lambda I_{\mathcal{A}} & A \\ A^\dagger & \lambda I_{\mathcal{A}} \end{bmatrix} : A \in S,\ \lambda \in \mathbf{C} \} .$$

Then $S_2$ is an operator system of $M_2(\mathcal{A})$. Define $\Psi : S_2 \to \mathcal{B} = M_2(\mathcal{B}(\mathcal{H}))$ by

$$\Psi \left( \begin{bmatrix} \lambda I_{\mathcal{A}} & A \\ A^\dagger & \lambda I_{\mathcal{A}} \end{bmatrix} \right) = \begin{bmatrix} \lambda I_{\mathcal{B}(\mathcal{H})} & \Phi(A) \\ \Phi(A)^\dagger & \lambda I_{\mathcal{B}(\mathcal{H})} \end{bmatrix} .$$

**Claim** $\Psi$ is completely positive on $S_2$. (Exercise)

Then, by Theorem 1.4, $\Psi$ can be extended to a completely positive map on $M_2(\mathcal{A})$. Hence, there exist linear maps $\Phi_{ij} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $1 \leq i, j \leq 2$, such that for $[A_{ij}] \in M_2(\mathcal{A})$, we have

$$\Psi \left( [A_{ij}] \right) = [\Phi_{ij}(A_{ij})]$$

Therefore, $\Phi_{12}$ is an extension of $\Phi$. Since $\Psi$ is completely positive, $\Psi(B)$ is self-adjoint for all self-adjoint $B \in M_2(\mathcal{A})$. In particular, for all $A \in \mathcal{A}$, we have

$$\begin{bmatrix} 0 & \Phi_{12}(A) \\ \Phi_{21}(A^\dagger) & 0 \end{bmatrix} = \Psi \left( \begin{bmatrix} 0 & A \\ A^\dagger & 0 \end{bmatrix} \right) = \Psi \left( \begin{bmatrix} 0 & A \\ A^\dagger & 0 \end{bmatrix} \right)^\dagger = \begin{bmatrix} 0 & \Phi_{21}(A^\dagger)^\dagger \\ \Phi_{12}(A)^\dagger & 0 \end{bmatrix} .$$

Therefore, $\Phi_{21}(A^\dagger) = \Phi_{12}(A)^\dagger$ for all $A \in \mathcal{A}$. Suppose $A \in \mathcal{A}$, with $\|A\| \leq 1$. Then we have

$$\begin{bmatrix} I_{\mathcal{A}} & A \\ A^\dagger & I_{\mathcal{A}} \end{bmatrix} \geq 0 \Rightarrow \Psi \left( \begin{bmatrix} I_{\mathcal{A}} & A \\ A^\dagger & I_{\mathcal{A}} \end{bmatrix} \right) = \begin{bmatrix} I_{\mathcal{B}(\mathcal{H})} & \Phi_{12}(A) \\ \Phi_{12}(A)^\dagger & I_{\mathcal{B}(\mathcal{H})} \end{bmatrix} \geq 0 \Rightarrow \|\Phi_{12}(A)\| \leq 1 .$$

This shows that $\Phi_{12}$ is a contraction. Similar argument applied to $I_k \otimes \Psi$ shows that $I_k \otimes \Phi_{12}$ is a contraction for all $k \geq 1$. Therefore, $\Phi_{12}$ is a complete contractive extension of $\Phi$ to $\mathcal{A}$. □

## 2 Completely positive linear maps on matrix spaces.

**Lemma 2.1** *Let $A \in M_n$. Then the following conditions are equivalent:*

1. *$\langle x|Ax\rangle \geq 0$ for all $|x\rangle \in \mathbf{C}^n$.*

2. *$A = A^\dagger$ and all eigenvalues of $A$ are non-negative.*

$A$ is said to be positive (semidefinite) ($A \geq 0$) if it satisfies any one of the above conditions. The set of all positive semidefinite matrices in $M_n$ will be denoted by $\mathcal{P}_n$.

**Definition 2.2** A linear map $\Phi : M_n \to M_m$ is *positive* ($\Phi \geq 0$) if $\Phi(\mathcal{P}_n) \subseteq \mathcal{P}_m$. More generally, given a positive integer $k$, define

$$\Phi_k = I_k \otimes \Phi : M_k(M_n) \cong M_{kn} \to M_k(M_m) \cong M_{km} \qquad \text{by} \qquad \Phi_k(A_{ij}) = (\Phi(A_{ij})).$$

$\Phi$ is *k-positive* if $\Phi_k$ is positive. Let $P_k(n,m)$ denote the set of $k$-positive map from $M_n$ to $M_m$. $\Phi$ is *completely positive* if $\Phi \in P_k(n,m)$ for all positive integer $k$.

Let $\{|e_i\rangle : 1 \leq i \leq n\}$ be the canonical basis for $\mathbf{C}^n$. Set $E_{ij} = |e_i\rangle\langle e_j|$. Then $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ is the canonical basis for $M_n$.

**Example 2.3** The map $\Phi : M_n \to M_n$ defined by $\Phi(A) = A^t$ is positive, but not 2-positive. To see this, consider $A = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \in M_{2n}$. Then $(I_2 \otimes \Phi)(A) = \begin{pmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{pmatrix}$ has a principal matrix of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is indefinite.

**Example 2.4** (Choi [3]) For every $n > 1$, the map $\Phi : M_n \to M_n$ with

$$\Phi(A) = (n-1)(\mathrm{Tr}A)I_n - A$$

is $n-1$-positive but not $n$-positive.

Given a linear map $\Phi : M_n \to M_m$, define the *Choi matrix* of $\Phi$ by

$$C(\Phi) = (\Phi(E_{ij}))_{i\,j=1}^n = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij}).$$

Theoretically, $\Phi$ is completely determined by its Choi matrix. In this section, we will explore the relationship between $\Phi$ and $C(\Phi)$. Let $S_k(n,m) = \{\sum_{i=1}^k |x_i\rangle|y_i\rangle : |x_i\rangle \in \mathbf{C}^n,\ |y_i\rangle \in \mathbf{C}^m\}$.

**Theorem 2.5** *Given a linear map $\Phi : M_n \to M_m$ and $k \geq 1$, the following conditions are equivalent:*

(a) *$\Phi$ is $k$-positive.*

(b) *$\langle z|C(\Phi)z\rangle \geq 0$ for all $|z\rangle \in S_k(n,m)$.*

(c) *$(I_n \otimes P)C(\Phi)(I_n \otimes P)$ is positive for all orthogonal projection $P$ with rank $\leq k$.*

*Proof.* (a) $\Leftrightarrow$ (b) : First consider the case $k = 1$. We have

$$
\begin{aligned}
& \Phi \geq 0 \\
\Leftrightarrow \quad & \Phi(|x\rangle\langle x|) \geq 0 \text{ for all } |x\rangle \in \mathbf{C}^n \\
\Leftrightarrow \quad & (|x\rangle \otimes I_m)^\dagger C(\Phi)(|x\rangle \otimes I_m) \geq 0 \text{ for all } |x\rangle \in \mathbf{C}^n \\
\Leftrightarrow \quad & \langle y| \Big( (|x\rangle \otimes I_m)^\dagger C(\Phi)(|x\rangle \otimes I_m) \Big) |y\rangle \geq 0 \text{ for all } |x\rangle \in \mathbf{C}^n, \ |y\rangle \in \mathbf{C}^m \\
\Leftrightarrow \quad & \langle y|\langle x|C(\Phi)|x\rangle|y\rangle \geq 0 \text{ for all } |x\rangle \in \mathbf{C}^n, \ |y\rangle \in \mathbf{C}^m \\
\Leftrightarrow \quad & \langle z|C(\Phi)z\rangle \geq 0 \text{ for all } |z\rangle \in S_1
\end{aligned}
$$

For general $k > 1$, by definition, $\Phi$ is $k$-positive if and only if $I_k \otimes \Phi$ is positive. We can apply the above result with $\Phi$ replaced by $I_k \otimes \Phi$ to get the result (exercise).

(b) $\Leftrightarrow$ (c) : Suppose (c) holds. Given $|z\rangle = \sum_{p=1}^{k} |x_p\rangle|y_p\rangle \in S_k(n, m)$, where $|x_p\rangle \in \mathbf{C}^n, \ |y_p\rangle \in \mathbf{C}^m$, $1 \leq p \leq k$, let $P$ be the orthogonal projection to the subspace spanned by $\{|y_p\rangle : 1 \leq p \leq k\}$. Then $(I_n \otimes P)(|z\rangle) = |z\rangle$. Therefore,

$$\langle z|C(\Phi)z\rangle = \langle (I_n \otimes P)z|C(\Phi)((I_n \otimes P)|z\rangle = \langle z|(I_n \otimes P)C(\Phi)(I_n \otimes P)z\rangle \geq 0$$

Conversely, suppose (b) holds. Let $P$ be an orthogonal projection in $M_m$ with rank $k$. Let $\{|y_p\rangle : 1 \leq p \leq k\}$ be an orthonormal basis of the range space of $P$. For every $|z\rangle \in \mathbf{C}^{mn}$, there exist $|x_p\rangle \in \mathbf{C}^n$, $1 \leq p \leq k$ such that $(I_k \otimes P)|z\rangle = \sum_{p=1}^{k} |x_p\rangle|y_p\rangle \in S_k$. We have

$$\langle z|(I_k \otimes P)C(\Phi)(I_k \otimes P)z\rangle = \langle (I_k \otimes P)z|C(\Phi)((I_k \otimes P)|z\rangle \geq 0$$

Hence, $(I_k \otimes P)C(\Phi)(I_k \otimes P) \geq 0$.

□

We have the following result by Choi [4] (see also [9]).

**Theorem 2.6** *Let $\Phi : M_n \to M_m$. The following are equivalent.*

(a) $\Phi$ *is completely positive.*

(b) $\Phi$ *is $n$-positive.*

(c) *The Choi matrix $C(\Phi) = (\Phi(E_{ij}))$ is positive.*

(d) $\Phi$ *admits an operator-sum representation:*

$$\Phi(A) \mapsto \sum_{j=1}^{r} F_j A F_j^\dagger. \tag{1}$$

*Furthermore, suppose (d) holds. Then we have*

(1) *The map* $\Phi$ *is unital if and only if* $\sum_{j=1}^{r} F_j F_j^\dagger = I_m$.

(2) *The map* $\Phi$ *is trace preserving if and only if* $\sum_{j=1}^{r} F_j^\dagger F_j = I_n$.

*Proof.* The equivalence of (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) follows from Theorem 2.5.

Suppose (c) holds. There exist $|f_j\rangle \in \mathbf{C}^{mn}$, $1 \le j \le r$ such that

$$(\Phi(E_{ij})) = |f_1\rangle\langle f_1| + \cdots + |f_r\rangle\langle f_r|.$$

Arrange each $|f_j\rangle$ as $m \times n$ matrix $F_j$ by taking the first $m$ entries as the first column, the next $m$ entries as the second column, etc., for each $j$. Once can check that

$$\Phi(E_{pq}) = \sum_{j=1}^{r} F_j E_{pq} F_j^\dagger.$$

By linearity, condition (d) holds.

Suppose (d) holds. Then

$$\Phi_k(A_{ij}) = \sum_{j=1}^{r} (I_k \otimes F_j)(A_{ij})(I_k \otimes F_j)^\dagger$$

is a positive map for each $k$. Thus (a) holds.

Furthermore, suppose (d) holds. We have

(1) $\Phi$ is unital $\Leftrightarrow \Phi(I_n) = I_m \Leftrightarrow \sum_{j=1}^{r} F_j F_j^\dagger = I_m$.

(2) $\Phi$ is trace preserving if and only if

$$\begin{aligned}
& \mathrm{Tr}\left(\sum_{j=1}^{r} F_j A F_j^\dagger\right) = \mathrm{Tr}A \text{ for all } A \in M_n \\
\Leftrightarrow\quad & \mathrm{Tr}\left(\sum_{j=1}^{r} A F_j^\dagger F_j\right) = \mathrm{Tr}A \text{ for all } A \in M_n \\
\Leftrightarrow\quad & \mathrm{Tr}\left(A\left(\sum_{j=1}^{r} F_j^\dagger F_j - I_n\right)\right) = 0 \text{ for all } A \in M_n \\
\Leftrightarrow\quad & \left(\sum_{j=1}^{r} F_j^\dagger F_j - I_n\right) = 0 \\
\Leftrightarrow\quad & \left(\sum_{j=1}^{r} F_j^\dagger F_j\right) = I_n
\end{aligned}$$

$\square$

**Remark 2.7**

- From mathematical point of view, checking the positivity of the Choi matrix $(\Phi(E_{ij}))$ is the simplest and most natural thing to do, and it turns out to be the right test for complete positivity.
- The matrices $F_j$ in (1) are called the *Choi operators* or the *Kraus operators*. In the context of quantum error correction, $F_j$ are called the *error operators*.

**Theorem 2.8** *Suppose $\Phi : M_n \to M_m$ of the form* (1) *such that $\{F_1, \ldots, F_r\}$ is linearly independent, and $\Phi$ has another operator sum representation*

$$\Phi(A) = \sum_{j=1}^{s} \tilde{F}_j A \tilde{F}_j^\dagger.$$

*Then there is $V = (v_{ij}) \in M_{r,s}$ such that $VV^*$
dag $= I_r$ and*

$$\tilde{F}_i = \sum_j v_{ij} F_j, \quad i = 1, \ldots, r.$$

*Proof.* Use the vector $f_j$ corresponding to $F_j$ to write

$$(\Phi(E_{ij})) = [|f_1\rangle \cdots |f_r\rangle][|f_1\rangle \cdots |f_r\rangle]^\dagger.$$

Also, use the vector $\tilde{f}_j$ corresponding to $\tilde{F}_j$ to write

$$(\Phi(E_{ij})) = [|\tilde{f}_1\rangle \cdots |\tilde{f}_s\rangle][|\tilde{f}_1\rangle \cdots |\tilde{f}_r\rangle]^\dagger.$$

Since $[|f_1\rangle \cdots |f_r\rangle]$ has linearly independent columns spending the column space of $(\Phi(E_{ij}))$, we see that

$$[|\tilde{f}_1\rangle \cdots |\tilde{f}_s\rangle] = [|f_1\rangle \cdots |f_r\rangle]V$$

for some $r \times s$ matrix $V = (v_{ij})$ such that $VV^\dagger = I_r$. □

Define an inner product on $M_{p,q}$ by $\langle X|Y\rangle = \mathrm{Tr}(X^\dagger Y)$. Suppose $\Phi : M_n \to M_m$ is a linear map. Then the dual map $\Phi^\dagger : M_m \to M_n$ is the linear map defined by

$$\langle B|\Phi^\dagger(A)\rangle = \langle \Phi(B)|A\rangle$$

for all $A \in M_m$ and $B \in M_n$.

**Theorem 2.9** *Let $\Phi : M_n \to M_m$. Then for every $k \geq 1$, $\Phi$ is $k$-positive if and only if $\Phi^\dagger$ is $k$-positive.*

*Proof.* Let $\{|e_i\rangle : 1 \le i \le n\}$ and $\{|f_s\rangle : 1 \le s \le m\}$ be canonical bases of $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively. Define $P : \mathbf{C}^{mn} \to \mathbf{C}^{mn}$ by $P(|f_s\rangle|e_i\rangle) = |e_i\rangle|f_s\rangle$ for all $1 \le i \le n$ and $1 \le s \le m$. Let $E_{ij} = |e_i\rangle\langle e_j|$ and $F_{st} = |f_s\rangle\langle f_t|$. We have

$$\begin{aligned}
& \langle e_i|\langle f_s|C(\Phi^\dagger)|f_t\rangle|e_j\rangle \\
= \ & \langle E_{ij}|\Phi^\dagger(F_{st})\rangle \\
= \ & \langle \Phi(E_{ij})|F_{st}\rangle \\
= \ & \overline{\langle F_{st}|\Phi(E_{ij})\rangle} \\
= \ & \overline{\langle f_s|\langle e_i|C(\Phi)|e_j\rangle|f_t\rangle} \\
= \ & \overline{(P(|f_s\rangle|e_i\rangle))^\dagger|C(\Phi)P|f_t\rangle|e_j\rangle} \\
= \ & \overline{\langle e_i|\langle f_s|P^\dagger C(\Phi)P|f_t\rangle|e_j\rangle}
\end{aligned}$$

Therefore, $C(\Phi^\dagger) = \overline{P^\dagger C(\Phi)P}$. Then we can apply Theorem 2.5 (b).

**Theorem 2.10** *Suppose $\Phi : M_n \to M_m$ is a completely positive linear map with operator sum representation in (1). Then the dual linear map $\Phi^\dagger : M_m \to M_n$ is by*

$$\Phi^\dagger(B) = \sum_{j=1}^{r} F_j^\dagger B F_j \,.$$

*Consequently, $\Phi^\dagger$ is also completely positive. Furthermore, $\Phi$ is unital (trace preserving, respectively) if and only if $\Phi^\dagger$ is trace preserving (unital, respectively).*

**Theorem 2.11** *Let $\Phi : M_n \to M_m$ be a linear map. Then $\Phi$ is a completely positive if and only if $\Phi$ is $k$-positive for $k = \min\{m, n\}$. In particular, if $n$ or $m$ equals to 1, then $\Phi$ is positive if and only if $\Phi$ is completely positive.*

*Proof.* It suffices to prove that if $m < n$ and $\Phi$ is $m$-positive, then $\Phi$ is completely positive.

By Theorem 2.9, if $\Phi$ is $m$-positive then $\Phi^\dagger$ is $m$-positive. By Theorem 2.6, $\Phi^\dagger$ is completely positive. Therefore, $\Phi$ is completely positive by Theorem 2.10. □

## 3 Cones of positive maps and duality

Given an inner product space $\mathcal{V}$, a non-empty subset $\mathcal{C}$ of $\mathcal{V}$ is said to be a *cone* if it satisfies:

1. $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$,

2. $r\mathcal{C} \subseteq \mathcal{C}$ for all $r \ge 0$,

$\mathcal{C}$ is said to be *pointed* if $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$ and *full* if $\mathcal{C} - \mathcal{C} = \mathcal{V}$.

Given a subset $S \subset \mathcal{V}$, define the *dual cone* of $S$ in $\mathcal{V}$ is given by

$$S^* = \{|v\rangle \in \mathcal{V} : \langle x|v\rangle \geq 0 \quad \text{for all } |x\rangle \in \mathcal{V}\}.$$

The following result describes some basic relationship between cone and dual cone.

**Theorem 3.1** *Suppose $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ are cones of $\mathcal{V}$. We have*

1. *$\mathcal{C}^*$ is a closed cone of $\mathcal{V}$.*
2. *$\mathcal{C}^*$ is pointed (full) if and only if $\mathcal{C}$ is full (pointed, respectively).*
3. *$\mathcal{C} \subseteq (\mathcal{C}^*)^*$. $\mathcal{C} = (\mathcal{C}^*)^*$ if and only if $\mathcal{C}$ is closed.*
4. *If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\mathcal{C}_1^* \supseteq \mathcal{C}_2^*$.*
5. *$(\mathcal{C}_1 \cap \mathcal{C}_2)^* \supseteq \mathcal{C}_1^* + \mathcal{C}_2^*$, and $(\mathcal{C}_1 \cap \mathcal{C}_2)^* = \mathcal{C}_1^* + \mathcal{C}_2^*$, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed.*
6. *$(\mathcal{C}_1 + \mathcal{C}_2)^* = \mathcal{C}_1^* \cap \mathcal{C}_2^*$.*

We will use this concept to reformulate the results in Theorems 2.5 and 5.1. See [19] for details. Recall from Section 2, the set

$$S_k(n,m) = \{\sum_{i=1}^{k} |x_i\rangle|y_i\rangle : |x_i\rangle \in \mathbf{C}^n, \ |y_i\rangle \in \mathbf{C}^m\}.$$

A Hermitian matrix $A \in M_n \otimes M_m$ is *$k$-block positive* if

$$\langle u|Au\rangle \geq 0 \text{ for all } |u\rangle \in S_k(m,n).$$

We will denote the $k$-block positive matrices in $M_n \otimes M_m$ by $BP_k(n,m)$. By Theorem 2.5, a linear map $\Phi$ is $k$-positive if and only if $C(\Phi) \in BP_k(n,m)$.

Let

$$Ent_k(n,m) = \{X \in M_n \otimes M_m : \langle A|X\rangle \geq 0 \text{ for all } \ A \in BP_k(n,m)\}$$

be the dual cone of the $BP_k(n,m)$. A density matrix is said to be $k$-entangled if it is in $Ent_k(n,m)$. 1-entangled states are *separable* (see Section 5).

The $k$-superpositive maps are given by

$$SP_k(n,m) = \{\Phi : M_n \to M_m : C(\Phi) \in Ent(n,m)\}.$$

We can show that (exercise) $\Phi \in SP_k(n,m)$ if and only if there exist $F_i \in M_{m,n}$ with rank $F_i \leq k$, $1 \leq i \leq r$ such that

$$\Phi(X) = \sum_{i=1}^{r} F_i X F_i^\dagger \quad \text{for all } X \in M_n.$$

Let $d = \min(n, m)$. For convenience of notation, we have omitted $(n, m)$ in the following

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Phi$ | $SP_1$ | $\subseteq$ | $SP_2$ | $\subseteq$ | $\cdots$ | $SP_d = CP = P_d$ | $\subseteq$ | $P_{d-1}$ | $\subseteq$ | $\cdots$ | $P_1$ |
| $\updownarrow$ | | | | | | | | | | | |
| $C(\Phi)$ | $Ent_1$ | $\subseteq$ | $Ent_2$ | $\subseteq$ | $\cdots$ | $Ent_d = P = BP_d$ | $\subseteq$ | $BP_{d-1}$ | $\subseteq$ | $\cdots$ | $BP_1$ |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{C}$ | $SP_1$ | $\subseteq$ | $SP_2$ | $\subseteq$ | $\cdots$ | $SP_d = CP$ | $Ent_1$ | $\subseteq$ | $Ent_2$ | $\subseteq$ | $\cdots$ | $Ent_d = P$ |
| $\updownarrow$ | | | | | | | | | | | | |
| $\mathcal{C}^*$ | $P_1$ | $\supseteq$ | $P_2$ | $\supseteq$ | $\cdots$ | $P_d = CP$ | $BP_1$ | $\supseteq$ | $BP_2$ | $\supseteq$ | $\cdots$ | $BP_d = P$ |

Under the correspondence $\Phi \leftrightarrow C(\Phi)$ and duality $\mathcal{C} \leftrightarrow \mathcal{C}^*$, study in one category, e.g. entanglement can be translated/related to study in other categories, e.g. positive maps, block positive matrices.

# 4 Decomposable positive linear Maps

Suppose $\Phi : M_n \to M_m$. For each $k$, define $\Phi_k^T : M_k \otimes M_n \to M_k \otimes M_m$ by $\Phi_k^T(A \otimes B) = A^T \otimes \Phi(B)$, and extend by linearity. $\Phi$ is said to be *$k$-copositive* if the map $\Phi_k^T$ is positive. $\Phi$ is *completely copositive* if $\Phi_k^T$ is positive for all $k$. $\Phi$ is *decomposable* if $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1$ is completely positive and $\Phi_2$ is completely copositive.

Suppose $n,\ m > 1$. If $n+m \leq 5$, every positive $\Phi : M_n \to M_m$ is decomposable. For $n+m > 5$, there exists a positive $\Phi : M_n \to M_m$ which is not decomposable.

The following criterion for decomposable map is due to Størmer [18]

**Theorem 4.1** *Let $\mathcal{A}$ be a $C^*$-algebra and $\Phi$ a linear map of $A$ into $\mathcal{B}(\mathcal{H})$. Then $\Phi$ is decomposable if and only if for all $k \geq 1$, whenever $(A_{ij})$ and $(A_{ji})$ belong to $M_k(\mathcal{A})^+$ then $(\Phi(A_{ij})) \in M_k(\mathcal{B}(\mathcal{H}))^+$.*

*Proof.* Suppose $\Phi$ is decomposable. Then $\Phi = \Theta + \Psi$ where where $\Theta$ is completely positive and $\Psi$ is completely copositive. For $k \geq 1$, $(A_{ij})$ and $(A_{ji})$ belong to $M_k(\mathcal{A})^+$, $\Theta_k((A_{ij})) \geq 0$ and $\Psi_k((A_{ij})) = (\Psi(A_{ij})) = \Psi_k^T((A_{ji})) \geq 0$. Therefore, $\Phi_k((A_{ij})) \geq 0$.

Conversely, suppose for all $k \geq 1$, whenever $(A_{ij})$ and $(A_{ji})$ belong to $M_k(\mathcal{A})^+$ then $(\Phi(A_{ij})) \in M_k(\mathcal{B}(\mathcal{H}))^+$.

Without loss of generality, we may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. We can also assume that $I_{\mathcal{B}(\mathcal{K})} \in \mathcal{A}$. Fix an orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{K}$ and let the elements in $\mathcal{B}(\mathcal{K})$ be represented by $A = (a_{ij})$, with $a_{ij} = \langle e_i|Ae_j\rangle$. Then we can define the transpose in $\mathcal{B}(\mathcal{K})$, $A^T = (a_{ji})$. Let $S = \{ \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix} \in M_2(\mathcal{B}(\mathcal{K})) : A \in \mathcal{A}\}$. Then $S$ is an operator system in $M_2(\mathcal{B}(\mathcal{K}))$. Define $\Psi : S \to \mathcal{B}(\mathcal{H})\}$ by

$$\Psi\left(\begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}\right) = \Phi(A).$$

For each $k \geq 1$, suppose $\left(A_{ij} \oplus A_{ij}^T\right) \in M_k(S)$ is positive. Then $(A_{ij})$, $\left(A_{ij}^T\right) \geq 0$. Since

$$\left(A_{ij}^T\right) \geq 0 \Rightarrow (A_{ji}) = \left(A_{ij}^T\right)^T \geq 0,$$

we have $\left(\Psi\left(A_{ij} \oplus A_{ij}^T\right)\right) = (\Phi(A_{ij})) \geq 0$. So, $\Psi$ is $k$-positive. Hence, $\Psi$ is completely positive on the operator system $S$. By Theorem 1.4, $\Psi$ can be extended to a completely positive map on $M_2(\mathcal{B}(\mathcal{H}))$. Let $\Theta_1,\ \Theta_2 : \mathcal{A} \to M_2(\mathcal{B}(\mathcal{K}))$ be given by

$$\Theta_1(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \Theta_2(A) = \begin{bmatrix} 0 & 0 \\ 0 & A^T \end{bmatrix}.$$

Then $\Theta_1$ is completely positive and $\Theta_2$ is completely copositive. Therefore, $\Phi = \Psi \circ (\Theta_1 + \Theta_2)$ is decomposable. □

**Example 4.2** (Choi [5]) Let $\Phi : M_3 \to M_3$ be given by

$$\Phi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = \begin{bmatrix} a_{11}+2a_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22}+2a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33}+2a_{22} \end{bmatrix}.$$

Then $\Phi$ is positive but not indecomposable.

*Proof.* To prove that $\Phi$ is positive, we use Theorem 2.5 (a) $\Leftrightarrow$ (c) for $k = 1$. By direct calculation,

$$C(\Phi) = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Suppose $P = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \overline{x_1} & \overline{x_2} & \overline{x_3} \end{bmatrix}$ is a rank one orthogonal projection. Then

$$(I_3 \otimes P)C(\Phi)(I_3 \otimes P) = X \otimes P, \text{ where } X = \begin{bmatrix} |x_1|^2+2|x_2|^2 & -\overline{x_1}x_2 & -\overline{x_1}x_3 \\ -x_1\overline{x_2} & |x_2|^2+2|x_3|^2 & -\overline{x_2}x_3 \\ -x_1\overline{x_3} & -x_2\overline{x_3} & |x_3|^2+2|x_1|^2 \end{bmatrix}.$$

Since

$$|x_1|^2 + 2|x_2|^2 \geq 0,$$

$$\det\left(\begin{bmatrix} |x_1|^2 + 2|x_2|^2 & -\overline{x_1}x_2 \\ \\ -x_1\overline{x_2} & |x_2|^2 + 2|x_3|^2 \end{bmatrix}\right) = 2(|x_2|^4 + |x_3|^3(|x_1|^2 + 2|x_2|^2)) \geq 0$$

$$\det(X) = 4\left(|x_1|^2|x_2|^4 + |x_1|^4|x_3|^2 + |x_1|^2|x_2|^2|x_3|^2 + |x_2|^2|x_3|^4\right) \geq 0\,,$$

we have $X \geq 0$. Hence, $(I_3 \otimes P)C(\Phi)(I_3 \otimes P) = X \otimes P \geq 0$. By Theorem 2.5, $\Phi$ is positive.

Next, we will use Theorem 4.1 to show that $\Phi$ is not decomposable.

Let $(x_{ij}) \in M_3(M_3)$ be given by

$$(x_{ij}) = \left[\begin{array}{ccc|ccc|ccc} 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \end{array}\right]$$

It is easy to check that $(x_{ij})$ and $(x_{ji})$ are positive but

$$\Phi((x_{ij})) = \left[\begin{array}{ccc|ccc|ccc} \mathbf{6} & 0 & 0 & 0 & \mathbf{-4} & 0 & 0 & 0 & \mathbf{-4} \\ 0 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 33 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 33 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{-4} & 0 & 0 & 0 & \mathbf{6} & 0 & 0 & 0 & \mathbf{-4} \\ 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 33 & 0 \\ \mathbf{-4} & 0 & 0 & 0 & \mathbf{-4} & 0 & 0 & 0 & \mathbf{6} \end{array}\right]$$

is not positive because $-2$ is an eigenvalue of $\Phi((x_{ij}))$. □

# 5 Completely positive map and entanglement

A positive semi-definite matrix $A \in M_n$ with $\mathrm{Tr}A = 1$ is called a *state (density matrix)*. A state $\rho \in M_n \otimes M_m \cong M_{nm}$ is said to be *separable* if there exist states $\rho_i^1 \in M_n$ and $\rho_i^2 \in M_m$, $i = 1, \ldots, k$ such that $\rho = \sum_{i=1}^k p_i \rho_i^1 \otimes \rho_i^2$ for some $p_i \geq 0$, $\sum_{i=1}^k p_i = 1$. Determining the separability of a state is one of the most important and difficult problems in quantum information. Suppose $\Phi : M_m \to M_n$ is a positive map. Then for all states $\rho^1 \in M_n$ and $\rho^2 \in M_m$, we have

$$(I_{M_n} \otimes \Phi)\,(\rho^1 \otimes \rho^2) = \rho^1 \otimes \Phi(\rho^2) \geq 0.$$

Hence, $\left(I_{M_n} \otimes \Phi\right)(\rho) \geq 0$ for all separable states $\rho$. The converse of this is also true and we have [7].

**Theorem 5.1** *A state $\rho \in M_{nm}$ is separable if and only if $\left(I_{M_n} \otimes \Phi\right)(\rho) \geq 0$ for all positive map $\Phi : M_m \to M_n$.*

The proof is divided into several lemmas.

**Lemma 5.2** *$\Phi : M_m \to M_m$ is positive if and only if $\operatorname{Tr}(C(\Phi)(P \otimes Q)) \geq 0$ for all orthogonal projections $P \in M_n$ and $Q \in M_m$.*

*Proof .*

$$\begin{aligned}
& \Phi \text{ is positive} \\
\Leftrightarrow\quad & \Phi(P) \text{ is a positive matrix in } M_m \text{ for all positive matrix in } M_n \\
\Leftrightarrow\quad & \Phi(P) \text{ is a positive matrix in } M_m \text{ for all rank one projection in } M_n \\
\Leftrightarrow\quad & (|x\rangle I_m)^\dagger C(\Phi)(|x\rangle \otimes I_m) \text{ is a positive matrix in } M_m \text{ for all } |x\rangle \in \mathbf{C}^n \\
\Leftrightarrow\quad & \langle y|(|x\rangle I_m)^\dagger C(\Phi)(|x\rangle \otimes I_m)|y\rangle \geq 0 \text{ for all } |x\rangle \in \mathbf{C}^n,\ |y\rangle \in \mathbf{C}^m \\
\Leftrightarrow\quad & \langle y|\langle x|C(\Phi)|x\rangle|y\rangle \geq 0 \text{ for all } |x\rangle \in \mathbf{C}^n,\ |y\rangle \in \mathbf{C}^m \\
\Leftrightarrow\quad & \operatorname{Tr}\left(C(\Phi)(|x\rangle|y\rangle\langle y|\langle x|)\right) \geq 0 \text{ for all } |x\rangle \in \mathbf{C}^n,\ |y\rangle \in \mathbf{C}^m \\
\Leftrightarrow\quad & \operatorname{Tr}\left(C(\Phi)(|x\rangle\langle x| \otimes |y\rangle\langle y|)\right) \geq 0 \text{ for all } |x\rangle \in \mathbf{C}^n,\ |y\rangle \in \mathbf{C}^m \\
\Leftrightarrow\quad & \operatorname{Tr}(C(\Phi)(P \otimes Q)) \geq 0 \ \text{ for all orthogonal projections } P \in M_n,\ Q \in M_m
\end{aligned}$$

**Lemma 5.3** *A state $\rho \in M_n \otimes M_m$ is separable if and only if $\operatorname{Tr}(\rho A) \geq 0$ for all $A \in M_{mn}$ such that $\operatorname{Tr}(A(P \otimes Q)) \geq 0$ for all orthogonal projections $P \in M_n$ and $Q \in M_m$.*

*Proof .* By definition, the set of separable states $\mathcal{S}$ is the convex hull of all $P \otimes Q$, where $P \in M_n$ and $Q \in M_m$ are rank 1 projections. The separation theorem says that $\rho \in M_n \otimes M_m$ is separable if and only if $f(\rho) \geq 0$ for all linear functional $f$ on $M_{mn}$, which is positive on $\mathcal{S}$. Since every linear functional $f$ on $M_{mn}$ is of the form $f(X) = \operatorname{Tr}(AX)$ for some $A \in M_{mn}$, the result follows.

**Proof of Theorem 5.1** Suppose a state $\rho \in M_n \otimes M_m$ such that $I_{M_n} \otimes \Phi(\rho) \geq 0$ for all positive map $\Phi : M_m \to M_n$. Let $A = (A_{ij}) \in M_m \otimes M_n$ such that $\operatorname{Tr}(A(P \otimes Q)) \geq 0$ for all orthogonal projections $P \in M_m$ and $Q \in M_n$. Choose $\Psi : M_n \to M_m$ such that $C(\Psi) = A$. Then by Lemma 5.2, $\Psi$ is positive. Hence, $\Phi = \Psi^\dagger : M_m \to M_n$ is also positive. Let $\{|e_i\rangle : 1 \leq i \leq n\}$ be the canonical basis for $\mathbf{C}^n$ and $E_{i\,j} = |e_i\rangle\langle e_j|$ . Then $\{E_{i\,j} : 1 \leq i, j \leq n\}$ is the set of canonical matrix units for $M_n$. We have

$$E = \sum_{i,j=1}^{n} E_{i\,j} \otimes E_{i\,j} = \left(\sum_{i=1} |e_i\rangle|e_i\rangle\right)\left(\sum_{j=1} |e_j\rangle|e_j\rangle\right)^\dagger$$

is positive and

$$C(\Phi) = (I_n \otimes \Phi)(E) .$$

Hence,

$$\begin{aligned}
& (I_n \otimes \Phi)(\rho) \geq 0 \\
\Rightarrow \quad & \langle E | (I_n \otimes \Phi)(\rho) \rangle \geq 0 \\
\Rightarrow \quad & \langle (I_n \otimes \Phi)^*(E) | \rho \rangle \geq 0 \\
\Rightarrow \quad & \langle (I_n \otimes \Psi)(E) | \rho \rangle \geq 0 \\
\Rightarrow \quad & \langle C(\Psi) | \rho \rangle \geq 0 \\
\Rightarrow \quad & \mathrm{Tr}\,(\rho A) \geq 0.
\end{aligned}$$

So, by lemma 5.3, $\rho$ is separable.

Define two partial transpose map on $M_n \otimes M_m$ by

$$T_1(A \otimes B) = A^T \otimes B, \quad \text{and} \quad T_2(A \otimes B) = A \otimes B^T$$

and extend by linearity. Note that for $(a_{ij}) \in M_n \otimes M_m$, we have

$$T_1((a_{ij})) = (a_{ji}), \quad \text{and} \quad T_2((a_{ij})) = (a_{ij}^T)$$

We have the PPT criterion for separability:

**Theorem 5.4** *(Horodecki [7]) Let $\rho$ be a state in $M_n \otimes M_m$. Then we have*

*(1) If $\rho$ is separable, then $T_2(\rho) \geq 0$.*

*(2) If $n + m \leq 5$ and $T_2(\rho) \geq 0$, then $\rho$ is separable.*

*Proof.* Note that $T_1(\rho) = (T_2(\rho))^T$. Therefore, the condition $T_2(\rho) \geq 0$ is equivalent to $T_1(\rho) \geq 0$. A state $\rho$ is said to be PPT if $T_2(\rho) \geq 0$.

(1) follows from Theorem 5.1 because the map $A \to A^T$ is positive.

To proof (2), suppose $n + m \leq 5$ and $T_2(\rho) \geq 0$. Let $\Phi : M_m \to M_n$ be a positive map. Then $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1 : M_m \to M_n$ is completely positive and $\Phi_2 : M_m \to M_n$ is completely copositive. Then $(I \otimes \Phi_1)(\rho) \geq 0$ and $(I \otimes \Phi_2)(\rho) = (I \otimes \Phi_2^T)(T_2(\rho)) \geq 0$. Hence, $(I \otimes \Phi)(\rho) \geq 0$. So, by Theorem 5.1, $\Phi$ is completely positive. □

To show that the conclusion in Theorem 5.4 (b) may not hold for $n = m = 3$, let $\Phi$ be as given in Example 4.2 and

$$\rho = \frac{1}{9}\left[\begin{array}{ccc|ccc|ccc} 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \end{array}\right]$$

Then by the discussion in Example 4.2, $\rho$, $T_2(\rho) \geq 0$. So, $\rho$ is a PPT state but $(I \otimes \Phi)(\rho) \not\geq 0$. Therefore, by Theorem 5.1, $\rho$ is not separable.

## 6 Interpolating problems

In this section, we study the following [11].

**Problem 6.1** *Given $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$, determine the necessary and sufficient condition for the existence of a completely positive linear map $\Phi : M_n \to M_m$ possibly with some special properties (e.g., $\Phi(I_n) = I_m$ or/and $\Phi$ is trace preserving) such that*

$$\Phi(A_j) = B_j \qquad \text{for } j = 1, \ldots, k. \tag{2}$$

Given $A = (a_{ij}) \in M_n$, let $\mathrm{vec}\,(A) = (a_{11}, \ldots, a_{1n}, \ldots, a_{21}, \ldots, a_{nn}) \in \mathbf{C}^{n^2}$. $A \to \mathrm{vec}\,(A)$ gives a linear isomorphism between $M_n$ and $\mathbf{C}^{n^2}$. Let $C = (C_{ij}) \in M_n(M_m)$, the *realignment* matrix of $C$ is given by

$$C^R = \begin{bmatrix} \mathrm{vec}\,(C_{11}) \\ \mathrm{vec}\,(C_{12}) \\ \vdots \\ \mathrm{vec}\,(C_{1n}) \\ \mathrm{vec}\,(C_{21}) \\ \vdots \\ \mathrm{vec}\,(C_{nn}). \end{bmatrix}$$

We have $\Phi(A) = \Phi(\sum_{i,j} a_{ij} E_{ij}) = \sum_{i,j} a_{ij} \Phi(E_{ij})$. Therefore,

$$\mathrm{vec}\,(\Phi(A)) = \mathrm{vec}\,(A) C(\Phi)^R \tag{3}$$

It follows from (3) that given $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$, (2) holds for some completely positive $\Phi$ if and only if there exists a positive semidefinite matrix $C \in M_{mn}$ such that

$$\mathrm{vec}\,(B_i) = \mathrm{vec}\,(A_i) C^R, \quad \text{for all } 1 \leq i \leq k \tag{4}$$

For general $A_i$ and $B_i$, checking if (4) holds for a positive semidefinite matrix $C \in M_{mn}$ can be very difficult. We will consider the case where $\{A_i : 1 \leq 1 \leq k\}$ and $\{B_i : 1 \leq 1 \leq k\}$ are commuting families of Hermitian matrices. In this case, there exist unitary matrices $U \in M_n$ and $V \in M_m$ such that $U^\dagger A_i U$ and $V^\dagger B_i V$ are diagonal matrices. Clearly, there is a completely positive map taking $A_i$ to $B_i$ if and only if there is a completely positive map taking $U^\dagger A_i U$ to $V^\dagger B_i V$. Therefore, we only need to consider the case where $A_i$, $B_i$ are diagonal matrices with diagonals $\mathbf{a}_i$, $\mathbf{b}_i$. In this case, $C$ can be chosen of the form $C = (C_{ij})$, where each $C_{ij}$ is an $m \times m$ diagonal matrix (exercise). So, we have

**Theorem 6.2** *Suppose $A_i$, $B_i$ are diagonal matrices with diagonals $\mathbf{a}_i$, $\mathbf{b}_i$. Then the following conditions are equivalent:*

1. *There exists a completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A_i) = B_i$ for all $1 \leq i \leq k$.*
2. *There exists an $n \times m$ nonnegative matrix $D$ such that $\mathbf{b}_i = \mathbf{a}_i D$ for all $1 \leq i \leq k$.*

A *nonnegative* matrix is *column* (respectively, *row*) *stochastic* if in each column (respectively, row) the entries sum up to 1. If $A$ is both column and row stochastic, then (the necessarily square matrix) $A$ is *doubly stochastic.*

**Theorem 6.3** $\Phi$ *in Theorem 6.2 can be choose to be unital (trace preserving, unital and trace-preserving, respectively) if and only if $D$ can be chosen to be column stochastic (row stochastic, doubly stochastic, respectively).*

Denote by $H_n$ the set of $n \times n$ Hermitian matrices. For $A \in H_n$, let

$$\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$$

be the vector of eigenvalues of $A$ with entries arranged in descending order.

**Corollary 6.4** *Let $A \in H_n$ and $B \in H_m$. Then the following conditions are equivalent.*

(a) *There is a completely positive linear map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$.*

(b) *There is a nonnegative $n \times m$ matrix $D$ such that $\lambda(B) = \lambda(A)D$.*

(c) *There are real numbers $\gamma_1, \gamma_2 \geq 0$ such that*

$$\gamma_1 \lambda_1(A) \geq \lambda_1(B) \quad \textit{and} \quad \lambda_m(B) \geq \gamma_2 \lambda_n(A).$$

*Proof.* The equivalence of (a) and (b) follows from Theorem 6.3.

(b) $\Rightarrow$ (c) : Suppose (b) holds with $D = (d_{ij})$. Let $\gamma_1 = \left(\sum_{i=1}^n d_{i1}\right)$ and $\gamma_2 = \left(\sum_{i=1}^n d_{im}\right) \lambda_1(A)$ Then we have

$$\lambda_1(B) = \sum_{i=1}^n d_{i1}\lambda_i(A) \le \left(\sum_{i=1}^n d_{i1}\right)\lambda_1(A) = \gamma_1\lambda_1(A) \text{ and}$$

$$\lambda_m(B) = \sum_{i=1}^n d_{im}\lambda_i(A) \ge \left(\sum_{i=1}^n d_{im}\right)\lambda_n(A) = \gamma_2\lambda_n(A)$$

(c) $\Rightarrow$ (b) : Suppose (c) holds. Then for each $1 \le j \le m$, we have

$$\gamma_1\lambda_1(A) \ge \lambda_1(B) \ge \lambda_j(B)\lambda_m(B) \ge \gamma_2\lambda_n(A).$$

So we can choose $0 \le t_j \le 1$ such that $\lambda_j(B) = t_j\gamma_1\lambda_1(A) + (1-t_j)\gamma_2\lambda_n(A)$. Let $D = (d_{ij})$ with

$$d_{ij} = \begin{cases} t_j\gamma_1 & \text{for } i = 1 \\ 0 & \text{for } 1 < i < n \\ (1-t_j)\gamma_2 & \text{for } i = n \end{cases}$$

Then we have $\lambda(B) = \lambda(A)D$. □

**Example 6.5** Let $A = \operatorname{diag}(2,1,0)$, $B_1 = \operatorname{diag}(4,3,1)$ and $B_2 = \operatorname{diag}(1,1,-1)$. There is a completely positive linear map $\Phi$ such that $\Phi(A) = B_1$, but there is no completely positive linear map $\Phi$ such that $\Phi(A) = B_2$.

Note that $(4,3,1) = (2,1,0)\begin{bmatrix} 2 & 3/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $B_1 = \sum_{j=1}^2 F_jAF_j^\dagger$ with

$$F_1 = \sqrt{2}E_{11} + \sqrt{3}E_{22}, \quad F_2 = E_{32}.$$

**Theorem 6.6** *Let $A \in H_n$ and $B \in H_m$. The following conditions are equivalent.*

(a) *There exists a unital completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$.*

(b) *There exists an $n \times m$ column stochastic matrix $D$ such that $\lambda(B) = \lambda(A)D$.*

(c) $\lambda_n(A) \le \lambda_i(B) \le \lambda_1(A)$ *for all* $1 \le i \le m$.

*Proof.* Similar to that of Theorem 2.1. Note that when $D$ is column stochastic, we can take $\gamma_1 = \gamma_2 = 1$. Conversely, if $\gamma_1 = \gamma_2 = 1$, the matrix $D$ constructed in the proof of 2.1 is column stochastic. □

**Theorem 6.7** *Suppose $(A,B) \in H_n \times H_m$. Denote by $\lambda_+(X)$ the sum of positive eigenvalues of a Hermitian matrix $X$. The following conditions are equivalent.*

(a) *There is a trace preserving completely positive map $\Phi : M_n \to M_m$ such that $\Phi(A) = B$.*

(b) *There exists an $n \times m$ row stochastic matrix $D$ such that $\lambda(B) = \lambda(A)D$.*

(c) $\lambda_+(B) \le \lambda_+(A)$, *and* $\mathrm{Tr}A = \mathrm{Tr}B$.

*Proof.* The equivalence of (a) and (b) follows from Theorem 6.3.

Let

$$\lambda_1(A) \ge \quad \cdots \ge \lambda_r(A) \ge 0 > \lambda_{s+1}(A) \ge \cdots \lambda_n(A)$$

$$\lambda_1(B) \ge \quad \cdots \ge \lambda_s(B) \ge 0 > \lambda_{r+1}(B) \ge \cdots \lambda_m(B).$$

(b) $\Rightarrow$ (c) : Suppose (b) holds with $D = (d_{ij})$. Then we have

$$\lambda_+(B) = \sum_{j=1}^{s}\sum_{i=1}^{n} d_{ij}\lambda_i(A) \le \sum_{j=1}^{s}\sum_{i=1}^{r} d_{ij}\lambda_i(A) \le \sum_{i=1}^{r}\sum_{j=1}^{m} d_{ij}\lambda_i(A) = \sum_{i=1}^{r}\lambda_i(A) = \lambda_+(A),$$

and

$$\mathrm{Tr}B = \sum_{j=1}^{m}\lambda_j(B) = \sum_{j=1}^{m}\sum_{i=1}^{n} d_{ij}\lambda_i(A) = \sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}\lambda_i(A) = \sum_{i=1}^{n}\lambda_i(A) = \mathrm{Tr}A.$$

(c) $\Rightarrow$ (b): Suppose $\mathrm{Tr}A = \mathrm{Tr}B$, $\lambda_+(A) \ge \lambda_+(B)$ Let $\lambda_-(A) = \mathrm{Tr}(A) - \lambda_+(A)$ and $\lambda_-(B) = \mathrm{Tr}(B) - \lambda_+(B)$. Then $\lambda_-(A) \le \lambda_-(B)$. Let

$$t_q = \begin{cases} \dfrac{\lambda_q(B)}{\lambda_+(A)} & \text{for } 1 \le q \le s, \\[2ex] \dfrac{\lambda_q(B)}{\lambda_-(A)} & \text{for } s < q \le m. \end{cases}$$

Here, if $\lambda_+(A) = 0$ then $\lambda_+(B) = 0$, and we can set $t_q = 0$ for $1 \le q \le s$. If $\lambda_-(A) = 0$ then $\lambda_-(B) = 0$, and $s = m$. Therefore, $t_q \ge 0$ for all $1 \le q \le m$. We have

$$\lambda_+(A) \ge \lambda_+(B) = (\lambda_+(A))\sum_{q=1}^{s} t_q, \quad \text{and} \quad |\lambda_-(A)| \ge |\lambda_-(B)| = |\lambda_-(A)|\sum_{q=s+1}^{n} t_q.$$

Let $u = 1 - \sum_{q=1}^{s} t_q \ge 0$, $v = 1 - \sum_{q=s+1}^{n} t_q \ge 0$ and $D$ be an $n \times m$ row stochastic matrix with

$$p\,\text{th row} = \begin{cases} (t_1, t_2, \dots, t_s, 0, \dots, 0, u) & \text{for } 1 \le p \le r, \\[2ex] (0, \dots, 0, t_{s+1}, t_{s+2}, \dots, t_{m-1}, t_m + v) & \text{for } r+1 \le p \le n\,. \end{cases}$$

We have ,

$$\begin{aligned} \textstyle\sum_{i=1}^{n} d_{ij}\lambda_i(A) = &\quad t_j\lambda_+(A) = \lambda_j(B), \quad \text{for } 1 \le j \le s \\ \textstyle\sum_{i=1}^{n} d_{ij}\lambda_i(A) = &\quad t_j\lambda_-(A) = \lambda_j(B), \quad \text{for } s+1 \le j < m \\ \textstyle\sum_{i=1}^{n} d_{im}\lambda_i(A) = &\quad u\lambda_+(A) + v\lambda_-(A) + t_m\lambda_-(A) = \lambda_m(B) \end{aligned}$$

because

$$u\lambda_+(A) + v\lambda_-(A) = (\lambda_+(A) - \lambda_+(B)) + (\lambda_-(A) - \lambda_-(B)) = 0.$$

Therefore, $\lambda(B) = \lambda(A)D$. □

**Example 6.8** Let $A = \mathrm{diag}\,(2, 1, -1)$, $B = \mathrm{diag}\,(2, 0, 0)$, and $C = \mathrm{diag}\,(1, 1, 0)$. Then there are trace preserving completely positive linear maps $\Phi_1, \Phi_2$ such that $\Phi_1(A) = B$, $\Phi_2(B) = C$, and $\Phi_2 \circ \Phi_1(A) = C$. There is no completely positive linear map $\Phi$ satisfying $\Phi(C) = A$.

**Remark 6.9** *For two density matrices $A$ and $B$, there is always a trace preserving completely positive map such that $\Phi(A) = B$. But there may not be a unital completely positive map $\Psi$ such that $\Psi(A) = B$.*

Suppose there is a unital completely positive map taking $A$ to $B$, and also a trace preserving completely positive map taking $A$ to $B$. Is there a unital trace preserving completely positive map sending $A$ to $B$? The following example shows that the answer is negative.

**Example 6.10** Suppose $A = \mathrm{diag}\,(4, 1, 1, 0)$ and $B = \mathrm{diag}\,(3, 3, 0, 0)$. By Theorems 6.6 and 6.7 there is a trace preserving completely positive map sending $A$ to $B$, and also a unital completely positive map sending $A$ to $B$. Let $A_1 = A - I_4 = \mathrm{diag}\,(3, 0, 0, -1)$ and $B_1 = B - I_4 = \mathrm{diag}\,(2, 2, -1, -1)$. By Theorem 6.7, there is no trace preserving completely positive linear map sending $A_1$ to $B_1$. Hence, there is no unital trace preserving completely positive map sending $A$ to $B$.

A quantum channel/completely positive map $\Phi : M_n \to M_n$ is called *mixed unitary* (mixing process) if there exist unitary matrices $U_1, \ldots, U_r \in M_n$ and positive numbers $p_1, \ldots, p_r$ summing up to 1 such that $\Phi(X) = \sum_{j=1}^{k} p_j U_j^\dagger X U_j$. Clearly, every mixed unitary completely positive map is unital and trace preserving. For $n \geq 3$, there exists a unital trace preserving completely positive map which is not mixed unitary [10].

For two real vectors $x, y$ of the same dimension, say, $n$, we say that $x$ is majorized by $y$, denoted by $x \prec y$, if the sum of entries of $x - y$ is zero, and the sum of the $k$ largest entries of $x$ is not larger than that of $y$ for $k = 1, \ldots, n - 1$; e.g., see [12] for the background on majorization.

**Example 6.11** $(3, 2, 1, 0) \prec (6, 1, 0, -1)$, $(3, 3, 0, 0) \not\prec (4, 1, 1, 0)$.

**Theorem 6.12** *Let $A, B \in H_n$. The following are equivalent.*

(a) *There exists a unital trace preserving completely positive map $\Phi$ such that $\Phi(A) = B$.*

(b) *There is a mixed unitary channel $\Phi$ such that $\Phi(A) = B$.*

(c) *There exist unitary matrices $U_j$, $1 \leq j \leq n$ such that $B = \frac{1}{n}\sum_{j=1}^{n} U_j A U_j^\dagger$.*

(d) *There is a unitary* $U$ *such that* $UAU^\dagger$ *has diagonal entries* $\lambda_1(B), \ldots, \lambda_n(B)$.

(e) $\lambda(B) \prec \lambda(A)$.

(f) *There is a doubly stochastic matrix* $D$ *such that* $\lambda(B) = \lambda(A)D$.

*Proof.* (a) $\Rightarrow$ (f) follows from Theorem 6.3

(f) $\Rightarrow$ (e) By the theory of majorization; see [12].

(e) $\Rightarrow$ (d): is Horn's Theorem; see [12]. (d) $\Rightarrow$ (c): Let $U \in M_n$ such that $UAU^\dagger$ with $\lambda_1(B), \ldots, \lambda_n(B)$ We may assume that $B = \mathrm{diag}\,(\lambda_1(B), \ldots, \lambda_n(B))$. Let $D = \mathrm{diag}\,(1, w, \ldots, w^{n-1})$ with $w = e^{i2\pi/n}$, and $F_j = D^j U$ for $j = 1, \ldots, n$. Then $B = (\sum_{j=1}^n F_j A F_j^\dagger)/n$.

The implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are clear.

□

**Remark 6.13**

- For non-commuting family, Problem 6.1 is very challenging and is still under current research. One needs to use other tools such as dilation, numerical range, etc. to study the problem.
- Also, there are study of Problem 6.1 with the extra restriction that $\Phi$ has a bounded Choi/Kraus rank. This corresponds to the problem of constructing the quantum operations with the minimum number of Choi/Kraus operators $F_1, \ldots, F_r$.

# 7 Exercises

1. Prove Stinespring's Dilation Theorem by the following steps: Define $\langle\ ,\ \rangle$ on the algebraic tensor product space $\mathcal{A}\otimes\mathcal{H}=\{\sum_{i=1}^n A_i\otimes\mathbf{x}_i : A_i\in\mathcal{A},\ \mathbf{x}_i\in\mathcal{H}\}$ by

$$\langle A\otimes\mathbf{x}, B\otimes\mathbf{y}\rangle = \langle\Phi(B^\dagger A)(\mathbf{x}),\mathbf{y}\rangle_{\mathcal{H}}$$

and extend by linearity. Use the completely positivity of $\Phi$ to show that for all $\mathbf{u}$, $\mathbf{v}\in\mathcal{A}\otimes\mathcal{H}$, we have

   (a) $\langle\mathbf{u},\mathbf{u}\rangle\geq 0$.

   (b) $\langle\mathbf{u},\mathbf{v}\rangle=\overline{\langle\mathbf{v},\mathbf{u}\rangle}$.

   (c) $|\langle\mathbf{u},\mathbf{v}\rangle|^2\leq\langle\mathbf{u},\mathbf{u}\rangle\langle\mathbf{v},\mathbf{v}\rangle$.

   (d) Let $\mathcal{N}=\{u\in\mathcal{A}\otimes\mathcal{H}:\langle\mathbf{u},\mathbf{u}\rangle=0\}$. Then show that $\mathcal{N}=\{u\in\mathcal{A}\otimes\mathcal{H}:\langle\mathbf{u},\mathbf{v}\rangle=0 \text{ for all } \mathbf{u}\in\mathcal{A}\otimes\mathcal{H}\}$ is a subspace of $\mathcal{A}\otimes\mathcal{H}$.

   (e) Use $\langle\mathbf{u}+\mathcal{N},\mathbf{v}+\mathcal{N}\rangle=\langle\mathbf{u},\mathbf{v}\rangle$ to define an inner product on the quotient space $(\mathcal{A}\otimes\mathcal{H})/\mathcal{N}$.

   (f) Let $\mathcal{K}$ be the Hilbert space completion of $(\mathcal{A}\otimes\mathcal{H})/\mathcal{N}$ under the inner product in 5. Let $A\in\mathcal{A}$. Show that the linear map $\pi(A):\mathcal{A}\otimes\mathcal{H}\to\mathcal{A}\otimes\mathcal{H}$ defined by

$$\pi(A)\left(\sum_{i=1}^n A_i\otimes\mathbf{x}_i\right)=\sum_{i=1}^n(AA_i)\otimes\mathbf{x}_i$$

   induces a bounded linear map on $\mathcal{K}$ such that $A\mapsto\pi(A)$ gives the required $*$-homorphism.

2. Let $C(T)$ be the commutative $C^*$-algebra of continuous function on the unit circle of the complex plane and $S$ the subspace of $C(T)$ spanned by $\{1,z,\overline{z}\}$. Define $\Phi:S\to M_2$ by

$$\Phi(a+bz+c\overline{z})=\begin{bmatrix} a & 2b\\ 2c & a\end{bmatrix}.$$

   Prove that $\Phi$ is positive.

3. Give an example of a norm-closed self-adjoint subspace $S$ of $M_2$ and a positive map from $S$ to $M_1$ which cannot be extended to a positive map from $M_2$ to $M_1$. (See also Theorem 1.5.)

4. Prove Lemma 1.7.

5. Prove the claim in the proof of Theorem 1.8.

6. Prove Lemma 2.1.

7. Suppose $\{E_{ij}:1\leq i,j\leq n\}$ is the set of canonical matrix units for $M_n$. Let $A=\sum_{i,j}^n E_{ij}\otimes E_{ji}$ and $B=\sum_{i,j}^n E_{ij}\otimes E_{ij}$. Prove that $A$ is unitary and $\frac{1}{n}B$ is an orthogonal rank one projection.

8. Let $A = (a_{ij}) \in M_n$. Define $\Phi_A : M_n \to M_n$ by $\Phi(B) = (a_{ij}b_{ij})$ for $B = (b_{ij}) \in M_n$. Prove that the following conditions are equivalent:

   (a) $\Phi_A$ is completely positive.

   (b) $\Phi_A$ is positive.

   (c) $A \geq 0$.

9. Given $\Phi_A : M_n \to M_m$, define $s_\Phi : M_m(M_n) \cong M_m \otimes M_n \to \mathbf{C}$ by $s_\Phi(F_{pq} \otimes E_{ij}) = (p, q)$ entry of $\Phi(E_{ij})$ and extend by linearity. Prove that $\Phi$ is completely positive if and only if $s_\Phi$ is positive.

10. Let $\Phi : M_n \to M_m$ be a linear map. Prove that there exist completely positive map $\Phi_j : M_n \to M_m$, $1 \leq j \leq 4$ such that $\Phi = \Phi_1 - \Phi_2 + i(\Phi_3 - \Phi_4)$.

11. Prove that the following two definitions for Schmidt rank of a unit vector $|u\rangle \in \mathbf{C}^n \otimes \mathbf{C}^m$ are equivalent.

    (a) The $k$ such that $|u\rangle = \sum_{i=1}^k s_i|x_i\rangle|y_i\rangle$, where $s_i > 0$, $1 \leq i \leq k$, $\{|x_i\rangle\}_{i=1}^k$ and $\{|y_i\rangle\}_{i=1}^k$ are orthonormal sets in $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively.

    (b) The smallest $k$ such that $|u\rangle \in S_k(n, m)$.

12. Write down the details for the proof of (a) $\Leftrightarrow$ (b) in Theorem 2.5 for $k > 1$.

13. Prove Theorems 2.10 and 2.11.

14. Prove Theorem 3.1.

15. Given $\Phi : M_n \to M_m$, define $\Psi^T : M_n \to M_m$ by $\Psi^T(A) = \Phi(A)^T$, the transpose of $\Phi(A)$. Prove that $\Phi$ is completely copositive if and only if $\Psi^T$ is completely positive.

16. Show that $\Phi \in SP_k(n, m)$ if and only if there exist $F_i \in M_{m,n}$ with rank $F_i \leq k$, $1 \leq i \leq r$ such that

$$\Phi(X) = \sum_{i=1}^r F_i X F_i^\dagger \quad \text{for all } X \in M_n\,.$$

17. Suppose $C = \in M_{nm})$ is positive senidefinite, $A_j \in M_n$ and $B_j \in M_m$ satisfy (4). Show that if $A_j$ and $B_j$ are diagonal matrices, then (4) still holds if we replace all off diagonal matrix of $C$ by 0.

18. Suppose $x = (x_1, \ldots, x_n)$ has nonnegative entries summing up to 1. Show that

$$(1, \ldots, 1)/n \prec x \prec (1, 0, \ldots, 0).$$

19. Show that conditions (a) – (d) in Theorem 6.7 is equivalent to the condition:

$$\mathrm{Tr}A = \mathrm{Tr}B \quad \text{and} \quad \sum_{j=1}^{n} |\lambda_j(A)| \geq \sum_{j=1}^{m} |\lambda_j(B)|.$$

20. Suppose $A,\ B \in H_n$. Prove that the following conditions are equivalent:

   (a) There is a unital trace preserving completely positive map such that $\Phi(A) = B$ if and only if the following holds.

   (b) For each $t \in \mathbf{R}$, there is a trace preserving completely positive map $\Phi_t$ such that

$$\Phi_t(A - tI) = B - tI.$$


## Acknowledgement

The author would like to thank the organizers of the Summer School on Quantum Information Science, Professor Jinchuan Hou, Professor Chi-Kwong Li and their colleagues at the Taiyuan University of Technology for their warm hospitality and support during his stay at Taiyuan. This work was also supported in part by a USA NSF grant.

## References


[1] W.B. Arveson, Subalgebras of $C^*$-algebras, Acta Math., 123 (1969).

[2] W.B. Arveson, Dilation theory yesterday and today. A glimpse at Hilbert space operators, 99123, Oper. Theory Adv. Appl., 207, Birkhuser Verlag, Basel, 2010.

[3] M.D. Choi, Positive linear maps on $C^*$-algebras, Canad. J. Math. 24 (1972), 520–529.

[4] M.D. Choi, Completely positive linear maps on complex matrices. Linear Algebra and Appl. 10 (1975), 285–290.

[5] M.D. Choi, Some assorted inequalities for positive linear maps on $C^*$-algebras, J. Operator Theory, 4 (1980), 271–285.

[6] M.D. Choi and E.G. Effros, Separable nuclear $C^*$-algebras and injectivity, Duke Math. J. 43 (1976), 309–322.

[7] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: necessary and sufficient conditions, Phys. Let. A, 223 (1996), 1–8.

[8] R. Kadison and J. Ringrose, Fundamentals of the theory of operator algebras I, Academic Press, 1983, New York.

[9] K. Kraus, States, effects, and operations: fundamental notions of quantum theory, Lectures in mathematics physics at the University of Texas at Austin, Lecture Notes in Physics 190, Springer-Verlag, Berlin-Heidelberg, 1983.

[10] L. Landau and R. Streater, On Birkhoffs theorem for doubly stochastic completely positive maps of matrix algebras, Linear Algebra and Appl., 193 (1993), 107–127.

[11] C.K. Li and Y.T. Poon, Interpolation by Completely Positive Maps, Linear and Multilinear Algebra, to appear.

[12] A.W. Marshall and I. Olkin, Inequaltiies: Theo Theory of Majorizations and its Applications, Academic Press, 1979.

[13] M. Nakahara and T. Ohmi, Quantum Computing: From Linear Algebra to Physical Realizations, CRS Press, New York, 2008.

[14] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, U.K., 2000.

[15] V. Paulsen, Completely Bounded Maps and Operator Algebras, Cambridge Studies in Advance Mathematics 18, Cambridge University Press, 2002.

[16] W.F. Stinespring, Positive functions on $C^*$-algebras, Pro. Amer. Math. Soc. 6 (1955), 211–216.

[17] E. Størmer, Positive linear map of operator algebras, Acta Math., 110 (1963), 233–278.

[18] E. Størmer, Decomposable positive linear maps on $C^*$-algebras, Proc. Amer. Math. Soc., 86 (1982), 402–404.

[19] Ł. Skowronek, E. Størmer and K. Życzkowski, Cones of positive maps and their duality relations, J. Math. Phy., 50 (2009), 062106.