An extension of the Fuglede-Putnam theorem to 
log-hyponormal operators $*^{† †}$

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Abstract

The familiar Fuglede-Putnam Theorem is as follows (see [5], [9] and [11]): If $A$ and $B$ are normal operators and if $X$ is an operator such that $AX = XB$, then $A^*X = XB^*$. In this paper, the hypothesis on $A$ and $B$ can be relaxed by using a Hilbert-Schmidt operator $X$: Let $A$ and $B^*$ be log-hyponormal operators such that $AX = XB$ for a Hilbert Schmidt operators $X$. Then $A^*X = XB^*$. As a consequence of this result, we obtain that the range of the generalized derivation induced by this class of operators is orthogonal to its kernel.

1 Introduction

Let $H$ be a separable infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all bounded operators on $H$. For any operator $A$ in $B(H)$ set, as usual, $|A| = (A^*A)^{1/2}$ and $[A^*, A] = A^*A - AA^*$ (the self commutator of $A$), and consider the following standard definitions: $A$ is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$, $p$-hyponormal if $(A^p - A^{*p}) \geq 0$. $A$ is said to be log-hyponormal if $A$ is invertible and satisfies the following equality

\[ \log(A^*A) \geq \log(TT^*). \]
It is known that invertible \( p \)-hyponormal operators are log-hyponormal operators but the converse is not true [18]. However it is very interesting that we may regards log-hyponormal operators as 0-hyponormal operators [18, 19]. The idea of log-hyponormal operator is due to Ando [3] and the first paper in which log-hyponormality appeared is [10]. See ([2, 18, 19, 21] for properties of log-hyponormal operators.

\( A \) is said to be \( p \)-quasihyponormal if \( A^* ((A^* A)^p - (AA^*)^p) A \geq 0 \) (\( p > 0 \)), \((p, k)\)-quasihyponormal if \( A^* ((A^* A)^p - (AA^*)^p) A^k \geq 0 \) \((p > 0, k \in \mathbb{N})\), if \( p = 1, k=1 \) and \( p = k = 1 \), then \( A \) is \( k \)-quasihyponormal, \( p \)-quasihyponormal and quasi-hyponormal respectively. \( A \) is normoloid if \( \|A\| = r(A) \) (the spectrum radius of \( A \)). Let \((N), (HN), Q(p), (Q(p, k))\) and \((NL)\) denote the classes constituting of normal, hyponormal, \( p \)-quasihyponormal, \( (p, k) \)-quasihyponormal, and normoloid operators. These classes are related by proper inclusion:

\[
(N) \subset (HN) \subset (Q(p)) \subset (Q(p, k)) \subset (NL).
\]

(see [12])

The familiar Fuglede-Putnam theorem is as follows (see [5], [9] and [11]):

**Theorem 1.1** If \( A \) and \( B \) are normal operators and if \( X \) is an operator such that \( AX = XB \), then \( A^* X = XB^* \).

S.K. Berberian [4] relaxes the hypothesis on \( A \) and \( B \) in Theorem 1.1 as the cost of requiring \( X \) to be Hilbert-Schmidt class. H.K. Cha [6] showed that the hyponormality in the result of Berberian [4] can be replaced by the quasihyponormality of \( A \) and \( B^* \) under some additional conditions. Lee ([13], Theorem 4) showed that the quasihyponormality in the above result can replaced by the \((p, k)\)-quasihyponormality of \( A \) and \( B^* \) with the additional condition \( \|A^{1-p}\| \|B^{-1}\|^{1-p} \leq 1 \). In [14] the author showed that Lee’s result remains true without the additional condition \( \|A^{1-p}\| \|B^{-1}\|^{1-p} \leq 1 \). In this paper we will show that the \((p, k)\)-quasihyponormality can be replaced by the log-hyponormality of \( A \) and \( B^* \). Let \( \delta_{A,B} \) be the generalized derivation defined on \( B(H) \) by \( \delta_{A,B}(X) = AX - XB \). As a consequence of this result, we obtain that the range of the generalized derivation induced by this class of operators is orthogonal to its kernel.

Let \( T \in B(H) \) be compact, and let \( s_1(X) \geq s_2(X) \geq ... \geq 0 \) denote the singular values of \( T \), i.e., the eigenvalues of \( |T| = (T^* T)^{1/2} \) arranged in their decreasing order. The operator \( T \) is said to belong to the Schatten \( p \)-class \( C_p \) if \( \|T\|_p = [\sum_{i=1}^{\infty} s_i(T)^p]^{1/p} = [tr|T|^p]^{1/p} < \infty, 1 \leq p < \infty \), where \( tr \) denotes the trace functional. Hence \( C_1(H) \) is the trace class, \( C_2(H) \) is the Hilbert-Schmidt class, and \( C_\infty \) is the class of compact operators with \( \|T\|_\infty = s_1(T) = \sup_{\|f\| = 1} \|Tf\| \) denoting the usual operator norm. For the general theory of the Schatten \( p \)-classes the reader is referred to [15], [16].
2 Main results

Lemma 2.1 Let $A$ and $B$ be operators in $B(H)$. If $A$ and $B^*$ are log-hyponormal operators such that $|A^*| \geq 1$ and $|B^*| \geq 1$, then the operator $\tau : C_2H \to C_2(H)$ defined by $\tau X = AXB$ is log-hyponormal.

**Proof.** It is known [4] that $\tau X = A^*XB^*$. Note that by the uniqueness of the square root of a positive operators we have

$$(\tau^\tau)^{\frac{1}{2}} X = |\tau| X = |A|X|B^*|, \quad (\tau^*\tau)^{\frac{1}{2}} X = |\tau^*| = |A^*|X|B|.$$

We also have

$$(\log |\tau| - \log |\tau^*|) X = \log |A|X\log |B^*| - \log |A^*|X\log |B|$$

$$= (\log |A| - \log |A^*|)X\log |B^*| + \log |A^*|X\log |B^*| - \log |B|). \quad (2.1)$$

The assumption $|A^*| \geq 1, |B^*| \geq 1$ implies $\log |A^*| \geq 0, \log |B^*| \geq 0$. Since $A$ and $B^*$ are log-hyponormal, $\log |A| - \log |A^*| \geq 0$ and $\log |B^*| - \log |B| \geq 0$. Hence,

$$\langle \log |\tau| - \log |\tau^*| \rangle X, X \rangle$$

$$= tr(X^{*}(\log |A| - \log |A^*|)X\log |B^*| + X^{*}\log |A^*|X(\log |B^*| - \log |B|)) \text{ by (2.1)}$$

$$= tr((\log |B^*| - \log |B|)^{\frac{1}{2}} X^{*}\log |A^*|X(\log |B^*| - \log |B|)^{\frac{1}{2}}) \geq 0.$$ 

Which completes the proof. \hfill \Box

Now we are ready to extend Putnam-Fuglede theorem to log-hyponormal operators.

**Theorem 2.1** Let $A$ and $B^*$ be log-hyponormal operators such that $|A^*| \geq 1$ and $|B^*| \geq 1$. If $AX = XB$ for $X \in C_2(H)$. Then $A^*X = XB^*$.

**Proof.** Recall that if $A$ is log-hyponormal, then the nonzero eigenvalues of $A$ are normal eigenvalues (i.e., if $\lambda \in \sigma_p(A) \setminus \{0\}$, then $\lambda \in \sigma_p(A^*)_*$)

Let $K$ be defined on $C_2(H)$ by $KY = AYB^{-1}$ for all $Y \in C_2(H)$. Since $B^*$ is log-hyponormal, $(B^*)^{-1}$ is also log-hyponormal. Then it follows from Lemma 2.1 that $K$ is invertible log-hyponormal, furthermore, $KX = AXB^{-1} = X$ and so, $X$ is an eigenvector of $K$. Now by applying (*) we get $K^*X = A^*X(B^{-1})^* = X$, that is, $A^*X = XB^*$ and the proof is achieved. \hfill \Box
Corollary 2.1 Let $A$ be log-hyponormal operator and $B^*$ be invertible $(p,k)$-quasihyponormal operators such that $|A^*| \geq 1$ and $|B^*| \geq 1$. If $AX = XB$ for $X \in C_2(H)$. Then $A^*X = XB^*$.

Proof. It is known that an invertible $(p,k)$-quasihyponormal operator is invertible $p$-hyponormal ([13], Lemma 3) and an invertible $p$-hyponormal is log-hyponormal [18]. Hence the result holds by the above theorem. \hfill \Box

Corollary 2.2 Let $A$ be invertible $(p,k)$-quasihyponormal and $B^*$ be log-hyponormal operators such that $|A^*| \geq 1$ and $|B^*| \geq 1$. If $AX = XB$ for $X \in C_2(H)$. Then $A^*X = XB^*$.

Theorem 2.2 Let $A, B$ be operators in $B(H)$ and $S \in C_2$. Then

\begin{equation}
\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2 \tag{2.1}
\end{equation}

and

\begin{equation}
\|\delta_{A,B}^*(X) + S\|_2^2 = \|\delta_{A,B}^*(X)\|_2^2 + \|S\|_2^2 \tag{2.2}
\end{equation}

if and only if $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$, for all $X \in C_2(H)$.

Proof. It is well known that the Hilbert-Schmidt class $C_2(H)$ is a Hilbert space under the inner product

$$\langle Y, Z \rangle = tr(Z^*Y) = tr(YZ^*).$$

Note that

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2 + 2Re(\delta_{A,B}(X), S)$$

$$= \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2 + 2Re(X, \delta_{A,B}(S))$$

and

$$\|\delta_{A,B}^*(X) + S\|_2^2 = \|\delta_{A,B}^*(X)\|_2^2 + \|S\|_2^2 + 2Re(X, \delta_{A,B}(S)).$$

Hence by the equality $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$ we obtain (2.1) and (2.2). \hfill \Box

Corollary 2.3 Let $A, B$ be operators in $B(H)$ and $S \in C_2$. Then

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2$$

and

$$\|\delta_{A,B}^*(X) + S\|_2^2 = \|\delta_{A,B}^*(X)\|_2^2 + \|S\|_2^2$$

if and only if $A$ and $B^*$ are log-hyponormal such that $|A^*| \geq 1$ and $|B^*| \geq 1$. 

References


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