A counterexample to several conjectures on Central Digraphs

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Abstract A counterexample is given for several conjectures on Central Digraphs, or, equivalently, about square zero-one matrices $A$ satisfying $A^2 = J$.

1 Introduction

A directed graph on $n$ vertices is called a Central Digraph if every two vertices are connected by a unique length two walk (there may be walks of other lengths). In terms of the adjacency matrix $A$ of the graph, this happens precisely when $A^2 = J_n$ where $J_n$ is the $n \times n$ matrix of all ones. In addition to the matrix and graph representations, these structures are in a one to one correspondence with an algebraic structure known as a Central Groupoid. Because of these connections, many researchers have studied this topic and obtained interesting results; see [2] and its references. Furthermore, researchers have made various conjectures on central groupoids. In this note, we give a counterexample to several of these conjectures.

2 Conjectures on Central Digraphs

Let $A_n$ be the set of all $n \times n$ matrices $A$, with $A^2 = J_n$. It is known (see [4]) that $A_n$ is non-empty if and only if $n = k^2$, for some $k \in \mathbb{N}$. The rows of $A$ can be partitioned into equivalence classes, where two rows are in the same equivalence class if they are identical. The multiset representing the number of rows in each equivalence class is called the Row Multiplicity of $A$. Column multiplicity can be defined in a similar way, where the equivalence classes are made up of identical columns of $A$ instead of rows. This leads to the following conjecture made in [3]:

Conjecture 2.1 (Multiplicity Conjecture) For all $n = k^2$, $k \in \mathbb{N}$, given any $A \in A_n$, the Row Multiplicity of $A$ is the same as the Column Multiplicity of $A$. 
In terms of the algebraic structure, one may wonder whether a large non-trivial subcentral groupoid can always be found in a given central groupoid. In matrix terms, we have the following conjecture in [1]:

Conjecture 2.2 (Sub-Groupoid Conjecture) Suppose \( k \in \mathbb{N} \) with \( k \geq 2 \). If \( A \in \mathcal{A}_{k^2} \), then \( A \) has a principal submatrix \( B \) such that \( B \in \mathcal{A}_{(k-1)^2} \).

To explain the next conjecture, we need more terminology. Suppose there exist, for some \( t \in \mathbb{N} \) integers \( i_1, \ldots, i_t, j_1, \ldots, j_t \) such that \( 1 \leq i_1 < i_2 \cdots < i_r \leq t \) and \( 1 \leq j_1 < j_2 \cdots < j_t \leq n \). Then denote by \( A[i_1, \ldots i_t; j_1 \ldots j_t] \) the \( t \times t \) submatrix of \( A \) consisting of rows \( i_1, \ldots, i_t \) and columns \( j_1, \ldots, j_t \) of \( A \). Then a Switch is defined as follows:

Definition Let \( A \in \mathcal{A}_n \), \( 1 \leq p < q \leq n \), \( 1 \leq r < s \leq n \) Suppose that \( \tilde{A} \) is obtainable by replacing \( A[p, q; r, s] \) with \( J_2 - A[p, q; r, s] \). If \( \tilde{A} \in \mathcal{A}_n \), then this replacement is called applying a “switch”, and it is said that \( \tilde{A} \) is obtainable by applying a switch to \( A \). If, for some \( A, B \in \mathcal{A}_n \), there exist \( t \in \mathbb{N} \) such that matrices \( A_0, A_1, \ldots, A_t \) exist, with \( A_0 = A \) and \( A_t = B \), with each \( A_i \in \mathcal{A}_n \) and each \( A_i \) obtainable by applying a switch to \( A_{i-1} \), then it is said that \( B \) is obtainable by applying a finite number of switches to \( A \).

In addition one needs the definition of the Standard Matrix in \( \mathcal{A}_n \). Denote by \( e_1, \ldots, e_k \) the standard basis for \( \mathbb{R}^k \). Let \( e = e_1 + \cdots + e_k \). Then one can easily describe the standard matrix \( A = (A_{ij})_{1 \leq i, j \leq k} \) in \( \mathcal{A}_n \), where again, \( n = k^2 \), and each \( A_{ij} \) is a \( k \times k \) matrix such that \( A_{ij} = e_j e_i^t \). It is elementary to check that \( A \) in fact is in \( \mathcal{A}_n \), and there are many reasons to consider this the standard matrix, one of which that it is the unique (up to permutation similarity) matrix of minimal rank \( \sqrt{n} \) in \( \mathcal{A}_n \) (this is proven in [2]). This leads to the third and final conjecture, made in [3]:

Conjecture 2.3 (Switch Conjecture) For all \( n = k^2 \), \( k \in \mathbb{N} \), given \( A \in \mathcal{A}_n \), it is possible to obtain \( A \) by applying a finite number of switches (and possibly permutations) to the standard matrix in \( \mathcal{A}_n \).

3 Auxiliary result and a counterexample

We now give a theorem, which is useful in disproving the above conjecture. This result can also be found in [1]. We include the proof of it for the sake of completeness.

Theorem 3.1 For any \( k \in \mathbb{N} \), with \( k \geq 2 \), given any \( A \in \mathcal{A}_{k^2} \), if \( A \) has a principal submatrix \( B \) with \( B \in \mathcal{A}_{(k-1)^2} \), then \( A \) contains at least \( k - 1 \) identical rows or at least \( k - 1 \) identical columns (or possibly both).

Proof. Assume such \( A \) and \( B \), satisfying the conditions of the theorem, exist. Then, let \( G \) be the directed graph with adjacency matrix \( A \), and let \( H \) be the induced subgraph of \( G \) such that the vertices of \( H \) correspond with the rows (and columns) of \( A \) which form \( B \) (so \( B \) is the adjacency matrix of \( H \)). Note that \( B \in \mathcal{A}_{(k-1)^2} \) implies that \( H \) is a central digraph.
To fix terminology, given any vertices \( x \) and \( y \) in \( G \), the phrases “\( x \) goes to \( y \)” and “\( y \) comes from \( x \)” both mean there is a directed edge \( x \rightarrow y \) in \( G \). It is known (see [4]) that a central digraph on \( n \) vertices has \( \sqrt{n} \) loops, and the adjacency matrix of such a central digraph has each row and column containing precisely \( \sqrt{n} \) ones. Thus, \( G \) contains \( k \) loops and \( H \) contains \( k - 1 \) loops, implying that there is precisely one loop in \( G \) is not in \( H \). Suppose \( x \) is the vertex of the loop. Now, assume that \( H \) contains some vertex \( a \) which goes to \( x \) and some vertex \( b \) which comes from \( x \). Then, \( a \rightarrow x \rightarrow b \) is the unique length two walk from \( a \) to \( b \) in \( G \), but \( x \) is not in \( H \), implying there is no length two walk from \( a \) to \( b \) in \( H \), a contradiction. Thus, \( H \) cannot contain both a vertex which goes to \( x \) and a vertex which comes from \( x \). So assume \( H \) contains no vertex that comes from \( x \). By above, the row corresponding to \( x \) in \( A \) has exactly \( k \) ones, so there are \( k \) vertices, including \( x \), which come from \( x \), so label the other \( k - 1 \) such vertices as \( y_1, y_2, \ldots, y_{k-1} \). Note that there can be no edge \( y_j \rightarrow y_i \) as then we would have \( x \rightarrow y_j \rightarrow y_i \) and \( x \rightarrow x \rightarrow y_i \) would be two length two walks from \( x \) to \( y_i \). So in particular, none of the \( y_j \) are loops. Also, there can be no edge \( y_j \rightarrow x \) as then we would have \( x \rightarrow x \rightarrow x \) and \( x \rightarrow y_j \rightarrow x \) as two length two walks from \( x \) to \( x \). Also, note that there can be no vertex \( z \) such that, for \( j \neq i \), both \( y_j \) and \( y_i \) go to \( z \), for if such a \( z \) existed, it would give two length two walks from \( x \) to \( z \) (\( x \rightarrow y_j \rightarrow z \) and \( x \rightarrow y_i \rightarrow z \)). Thus the vertices which each \( y_j \) go to are distinct, and as no \( y_j \) is a loop, there are \( k \) such vertices. Label the vertices that \( y_j \) go to as \( z_{j,1}, z_{j,2}, \ldots, z_{j,k} \). Note that this labels all \( k^2 \) vertices of \( G \). Further, for each \( y_j \), there is a unique length two walk from \( y_j \) to itself, and as \( y_j \) is not a loop, there is a unique \( z \) with \( y_j \rightarrow z \rightarrow y_j \). Relabel the \( z_{j,i} \), if necessary, so that \( z_{j,1} \) is the unique such vertex. Then, as \( z_{j,1} \rightarrow y_j \rightarrow z_{j,1} \) is the unique length two walk from \( z_{j,1} \) to itself, and \( y_j \) is not in \( H \) (by our earlier assumption), so thus \( z_{j,1} \) is not in \( H \) either (for if it were, there would be no length two walk from \( z_{j,1} \) to itself in \( H \)). Thus, as \( x, y_1, y_2, \ldots, y_{k-1}, z_{1,1}, z_{2,1}, \ldots, z_{k-1,1} \) represent \( 2k - 1 \) vertices in \( G \) which aren’t in \( H \), it follows all of the remaining vertices of \( G \) are in \( H \) (as \( G \) contains \( k^2 \) vertices and \( H \) contains \( (k - 1)^2 \) vertices). Further, assume there were some vertex \( w \) in \( H \) such that \( w \) goes to some \( y_j \). Note that by above, \( z_{j,2} \) is in \( H \), so there is some \( a \) in \( H \) with \( w \rightarrow a \rightarrow z_{j,2} \). Further, as \( y_j \) goes to \( z_{j,2} \), we have \( w \rightarrow y_j \rightarrow z_{j,2} \), and, as \( y_j \) is not in \( H \), \( y_j \neq a \), so we have two length two walks from \( y_j \) to \( z_{j,2} \) in \( G \), a contradiction. Thus, no vertex in \( H \) goes to \( y_j \). Thus, as \( y_i \) doesn’t go to \( y_j \) either, for any \( i \) (\( 1 \leq i \leq k - 1 \)), it follows the only vertices which can go to \( y_j \) are in the set \( S = \{ x, z_{1,1}, z_{2,1}, \ldots, z_{k-1,1} \} \). But, as \( |S| = k \) and there must be \( k \) vertices which go to \( y_j \), it follows that each vertex in \( S \) must go to \( y_j \). But this is independent of \( j \), so the set of vertices which go to \( y_j \) is the same for \( 1 \leq j \leq k - 1 \) which implies that, in \( A \), there are at least \( k - 1 \) identical columns (the columns corresponding to each \( y_j \) must all be identical).

So, the last possibility is that \( H \) contains some vertex that comes from \( x \). But then, by above, it must contain no vertex which goes to \( x \). So let \( G' \) and \( H' \) be the directed graphs which result from reversing each edge in \( G \). It is not hard to see that \( G' \) and \( H' \) both are central digraphs, with \( H' \) containing no vertex that comes from \( x \) in \( G' \) so by my above analysis, the adjacency matrix for \( G' \) contains at least \( k - 1 \) identical columns. However,
it is clear that the adjacency matrix for \( G' \) is \( A^T \), which implies \( A \) contains at least \( k - 1 \) identical rows.

Now, we are ready to present the following theorem showing that all the conjectures mentioned in Section 2 are false.

**Theorem 3.2** There exists a matrix \( M \in \mathcal{A}_{64} \) where the Row Multiplicity of \( M \) is not equal to the Column Multiplicity of \( M \), \( M \) contains no principal submatrix \( B \) with \( B \in \mathcal{A}_{49} \), and it is not possible to obtain \( M \) from applying a finite number of switches, and or permutations to the standard matrix in \( \mathcal{A}_{64} \).

**Proof.** Consider the following matrix \( M \): [One may down load the matrix from http://??? to check the calculation.]
First, it can be computationally verified that $M \in A_{64}$. It also is trivial to check that the Row Multiplicity of $M$ is different from the Column Multiplicity of $M$ (in particular $M$ has many pairs of identical rows, but no two distinct columns are identical). Further, it is clear from inspection that $M$ does not contain any 7 identical rows or 7 identical columns, so by Theorem 3.1, $M$ cannot contain any principal submatrix $B \in A_{49}$. Now, if $M$ were obtainable by applying a finite number of switches and or permutations to the standard matrix, this would imply the existence of a matrix $B \in A_{64}$ such that $M$ was obtainable by applying a switch to $B$. Equivalently, $B$ would be obtainable by applying a switch to $M$, since, if the switch that obtains $M$ is given by replacing $B[p, q; r, s]$ with $J_2 - B[p, q; r, s]$, then $J_2 - M[p, q; r, s] = B[p, q; r, s]$, so $B$ is then obtainable by applying that same particular switch to $M$. However, as is shown in [2], it is only possible to apply a switch to $M$, say by replacing $M[p, q; r, s]$ with $J_2 - M[p, q; r, s]$ to obtain some matrix $B \in A_{64}$ if, in particular, columns $p$ and $q$ are identical in $M$. However, by above, $M$ does not contain any two distinct columns which are identical, so it is not possible to apply such a switch to $M$. Thus, it is not possible to obtain $M$ by applying a finite number of switches and or permutations to the standard matrix in $A_{64}$. \[\square\]

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References


