Eigenvalues of the sum of matrices from unitary similarity orbits

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Based on some joint work with:
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**Basic problem**

Let $A, B \in M_n$. Determine the set $\mathcal{E}(A, B)$ of eigenvalues of matrices of the form

$$U^* AU + V^* BV, \quad U, V \text{ are unitary},$$
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- It is natural to make predictions about \( U^*AU + V^*BV \) based on information of \( A \) and \( B \).
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- Especially, in the study of quantum computing and quantum information theory, all measurements, control, perturbations, etc. are related to unitary similarity transforms.
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Just consider the eigenvalues of

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\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}
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with \( t \in [0, \pi] \).
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Example Let \( A = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix} \) and \( B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \).
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If $A + V^*BV$ has eigenvalues $c_1 \geq c_2$, then $c_1 \in [23, 24]$, $c_2 \in [13, 14]$, 
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If \( A + V^*BV \) has eigenvalues \( c_1 \geq c_2 \), then \( c_1 \in [23, 24], c_2 \in [13, 14] \), and \( \mathcal{E}(A, B) = [13, 14] \cup [23, 24] \).
Theorem

Let $A = \text{diag} (a_1, \ldots, a_n)$ and $B = \text{diag} (b_1, \ldots, b_n)$ with

$$a_1 \geq \cdots \geq a_n \quad \text{and} \quad b_1 \geq \cdots \geq b_n.$$
Results on Hermitian matrices

Theorem

Let \( A = \text{diag} (a_1, \ldots, a_n) \) and \( B = \text{diag} (b_1, \ldots, b_n) \) with
\[
a_1 \geq \cdots \geq a_n \quad \text{and} \quad b_1 \geq \cdots \geq b_n.
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If \( V \) is unitary and \( A + V^*BV \) has eigenvalues \( c_1 \geq \cdots \geq c_n \), then
\[
c_j = [b_j + a_n, b_j + a_1] \cap [a_j + b_n, a_j + b_1] \quad \text{for} \quad j = 1, \ldots, n.
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It follows that $\mathcal{E}(A, B)$ equals

$$[a_n + b_n, a_1 + b_1] \setminus \bigcup_{j=1}^{n-1} ((a_{j+1} + b_1, a_j + b_n) \cup (b_{j+1} + a_1, b_j + a_n)).$$
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$$[a_n + b_n, a_1 + b_1] \setminus \bigcup_{j=1}^{n-1} \left( (a_{j+1} + b_1, a_j + b_n) \cup (b_{j+1} + a_1, b_j + a_n) \right).$$

Consequently, $\mathcal{E}(A, B) = [a_n + b_n, a_1 + b_1]$ if

$$b_1 - b_n \geq \max_{1 \leq j \leq n-1} (a_j - a_{j+1}) \quad \text{and} \quad a_1 - a_n \geq \max_{1 \leq j \leq n-1} (b_j - b_{j+1}).$$
Theorem [Klychko, Fulton, A. Horn, Thompson, Kuntson, Tao, ... ]

Let $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$ and $c_1 \geq \cdots \geq c_n$ be given.
Theorem [Klychko, Fulton, A. Horn, Thompson, Kuntson, Tao, ... ]

Let \( a_1 \geq \cdots \geq a_n, \ b_1 \geq \cdots \geq b_n \) and \( c_1 \geq \cdots \geq c_n \) be given.

There exist Hermitian matrices \( A, B \) and \( C = A + B \) with eigenvalues \( a_1 \geq \cdots \geq a_n, \ b_1 \geq \cdots \geq b_n, \) and \( c_1 \geq \cdots \geq c_n \) if and only if
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$$\sum_{j=1}^{n} (a_j + b_j) = \sum_{j=1}^{n} c_j,$$

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$$\sum_{j=1}^{n}(a_j + b_j) = \sum_{j=1}^{n} c_j,$$

and

$$\sum_{r \in R} a_r + \sum_{s \in S} b_s \geq \sum_{t \in T} c_t$$

for all subsequences $R, S, T$ of $(1, \ldots, n)$ determined by the Littlewood-Richardson rules.
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Then $\mathcal{E}(A, B)$ equals
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Proposition [LPS, 2008]

Suppose $A, B \in M_n$ are normal with

$$\sigma(A) = \{a_1, a_2\} \quad \text{and} \quad \sigma(B) = \{b_1, b_2\}.$$ 

Then $\mathcal{E}(A, B)$ are two (finite) segments of the hyperbola with end points in $\{a_1 + b_1, a_1 + b_2, a_2 + b_1, a_2 + b_2\}$.
Example 3 Suppose $w = e^{i2\pi/3}$,
$\sigma(A) = \{-iw, -iw^2\}$ and $\sigma(B) = \{-i, -wi, -w^2i\}$. 
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Proposition [LPS,2008]

Suppose $\sigma(A) = \{a_1, a_2\}$ and $\sigma(B) = \{b_1, b_2, b_3\}$. Then $E(A, B) = E(a_1, a_2; b_1, b_2, b_3)$ consists of connected components enclosed by the three pairs of hyperbola segments

\[ E(a_1, a_2; b_1, b_2), \ E(a_1, a_2; b_1, b_3), \ E(a_1, a_2; b_2, b_3). \]
Example 6 Suppose $\sigma(A) = \{0, 1, 4, 6\}$ and $\sigma(B) = \{0, i, 2i\}$. 
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Results on normal matrices

Theorem [LPS,2008]

Suppose $A, B \in M_n$ are normal with $\sigma(A) = \{a_1, \ldots, a_p\}$ and $\sigma(B) = \{b_1, \ldots, b_q\}$. Then

$$\mathcal{E}(A, B) = (\cup \mathcal{E}(a_{i_1}, a_{i_2}, a_{i_3}; b_{j_1}, b_{j_2})) \cup (\cup \mathcal{E}(a_{i_1}, a_{i_2}; b_{j_1}, b_{j_2}, b_{j_3})),$$

where $\mathcal{E}$ denotes the eigenvalue set.
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**Theorem [Wielandt,1955], [LPS,2008]**

Suppose $A, B \in M_n$ are normal. Then $\mu \notin \mathcal{E}(A, B)$ if and only if there is a circular disk containing the eigenvalues of $A$ or $\mu I - B$, and excluding the eigenvalues of the other matrices.
The **Davis-Wielandt Shell** of $A \in M_n$ is the set

$$DW(A) = \{(x^* Ax, \|Ax\|^2) : x \in \mathbb{C}^n, x^*x = 1\}$$

$$\subseteq \{(z, r) \in \mathbb{C} \times \mathbb{R} : |z|^2 \leq r\}.$$
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- Then $\mu \in \sigma(A)$ if and only if $(\mu, |\mu|^2) \in DW(A)$. 

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**Proposition**

Let $A \in M_n$.

- Then $\mu \in \sigma(A)$ if and only if $(\mu, |\mu|^2) \in DW(A)$.
- Then $A$ is normal if and only if $DW(A)$ is a polyhedron.
Theorem [LPS,2008]

Suppose $A, B \in M_n$. Then $\mu \in \mathcal{E}(A, B)$ if and only if any one of the following holds.
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- $DW(A) \cap DW(\mu I - B) \neq \emptyset$. 

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- $DW(A) \cap DW(\mu I - B) \neq \emptyset$.
- For any $\xi \in \mathbb{C}$,

  $$\text{conv} \, \sigma(|A + \xi I|) \cap \text{conv} \, \sigma(|B - \xi I - \mu I|) \neq \emptyset.$$
Theorem [LPS,2008]

Suppose $A, B \in M_n$. Then $\mu \in \mathcal{E}(A, B)$ if and only if any one of the following holds.

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- For any $\xi \in \mathbb{C}$,

$$\text{conv} \sigma(|A + \xi I|) \cap \text{conv} \sigma(|B - \xi I - \mu I|) \neq \emptyset.$$ 

Equivalently, singular values of $A + \xi I$ and the singular values of $B - \xi I - \mu I$ do not lie in two separate closed intervals.
Further research

- Develop computer programs to generate $\mathcal{E}(A, B)$ for general $A, B \in M_n$. 
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- Determine all possible eigenvalues for $\sum_{j=1}^{k} U_j^*A_jU_j$ for given $A_1, \ldots, A_k \in M_n$. 
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- Study the above problems for unitary matrices chosen from a certain subgroups such as $SU(2) \otimes \cdots \otimes SU(2)$ ($m$ copies).
Thank you for your attention!