Problems for mathematical educations

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Problem Given n sticks of lengths $d_1, \ldots, d_n$.

(a) When can use them to form a convex polygon?

Answer: the longest stick has length not greater than the sum of the lengths of the other $n-1$ sticks.

(b) How to form the convex polygon with maximum area?

Answer: Form a polygon with vertices lying on a circle.
[One can show that it can always be done.]

Problem Given n sticks of lengths $d_1, \ldots, d_n$. If we are allow to add a stick of any length we like, say, d_{n+1}, how to form the convex polygon with maximum area with the sticks $d_1, \ldots, d_n, d_{n+1}$.

Answer: For a polygon so that all vertices are on the circle, and the additional stick of length d_{n+1} is the diameter.

Poon has prepare some very thorough note on these problems.

Problem Given an 8×8 chessboard, how many $k \times k$ squares are there?

Answer: $(n+1-k)^2$

Consequently, there are $1^2 + \cdots + 8^2$ so many squares on the chessboard.

Important idea: One can reduce the problem on 1 dimensional case.

Problem Given a $m \times n$ chessboards, how many rectangles are there.

Answer: $C(m+1, 2) \cdot C(n+1, 2) =$ number of ways to find left margins and left margins.

Other extensions.

1. Count rectangles with fixed ratio of width and length.
2. Count equilateral triangles with length k on each side in a equilateral triangle with length n on each side, where $1 \leq k \leq n$.
3. Extend to higher dimensions. For example, there are $1^3 + \cdots + 8^3$ so many cubes in an $8 \times 8 \times 8$ cubes.

Problem Given n unit cubes, how to put them together to get the minimum surface area?

Problem Given n unit tiles, how to put them together to get a region with minimum perimeter (to be painted)?

Answer: If $n = k^2$ make the $k \times k$ square; if $k^2 < n < (k+1)^2$ make the $k \times k$ square labeled by $(1,1),(1,2),\ldots,(k,k)$, and then add the remaining squares in the order of $(1,k+1),(2,k+1),\ldots,(k+1,k+1),(k+1,k),\ldots,(k+1,1)$ until the tiles are exhausted.

Proof. [An idea by Raymond Sze and Nam-Kiu Tsing.] Unless $n = 1$, in the remaining construction, there are 4 types of tiles according to how many of its 4 edges needed to be painted in the final construction. Suppose there are x_0, x_1, x_2, x_3 of them with $0,1,2,3$ exposed edges (to be painted). So, $x_0 + x_1 + x_2 + x_3 = n$, and we want to minimize $x_1 + 2x_2 + 3x_3$.

Clearly, we want to make x_0 as large as possible, x_3 as small as possible.

Suppose a given shape has minimum perimeter.

First we can align the tiles so that each tile has boundary in the vertical and horizontal positions.

Put the shape on a paper with squares of the same size as the tiles and horizontal and vertical edges labeled by $1,2,3,\ldots$ so that the left bottom square is the $(1,1)$ square.

Suppose there are m rows of tiles in the shape. We can slide all of them so that the left margin of each row are the same and tough the left edge of the paper without increasing the perimeter.

Similarly, we can slide all the column to touch the bottom edge of the paper without increasing the perimeter.

If the shape is not in the form we describe, we can move the boundary tiles in the East and North to fill the $k \times k$ left bottom squares without increasing the perimeter till we get the shape we describe.

[More details needed.]

Extend the result and proof to 3-dimensional problem.

Problem Given an 8×8 chessboard, one can use thirty two 1×2 dominoes to cover it. If one removes the two squares at the NW and SE corner, can we use thirty one 1×2 dominoes to cover it?

Answer: No. Because every domino over a red and white squares, but we have remove two black squares.

Other extensions

Punctured chessboard. Consider $m \times n$ chessboard with some squares deleted. [The solution depending on a perfect matching in some bipartite graph using the black squares and red squares as vertices, and two vertices are joined to each other if they are next to each other in the punctured chessboard.]

Other types of dominoes. [Inspired by Vladimir Bolotnikov.] Consider $m \times n$ chessboard.

Can we cover it with ”L” shape dominoes, i.e., $\begin{array}{cc} * & \\ * & \\ * & * \end{array}$.

How about T” shape dominoes?

Three dimensional version. For an $8 \times 8 \times 8$ cube we can decompose it into 128 $1 \times 1 \times 2$ blocks. If we remove two of the corners....