Some Exercises on Pandigital Numbers

Problem 1: Fill in the digits 1 to 9 (using every digit once) so that

$$\begin{array}{cccc} & \square & \square & \square \\ + & \square & \square & \square \\ \hline & \square & \square & \square \end{array}$$

For example,

$$\begin{array}{cccc} & \boxed{1} & \boxed{4} & \boxed{2} \\ + & \boxed{5} & \boxed{9} & \boxed{6} \\ \hline & \boxed{7} & \boxed{3} & \boxed{8} \end{array}$$

This problem can be made easier by assigning some digits:

1.

$$\begin{array}{cccc} & \boxed{1} & \square & \boxed{3} \\ + & \square & \boxed{8} & \square \\ \hline & \boxed{4} & \square & \boxed{9} \end{array}$$

2.

$$\begin{array}{cccc} & \boxed{1} & \square & \boxed{7} \\ + & \square & \boxed{5} & \square \\ \hline & \boxed{4} & \square & \boxed{6} \end{array}$$

3.

$$\begin{array}{cccc} & \boxed{2} & \square & \square \\ + & \square & \boxed{4} & \boxed{9} \\ \hline & \boxed{5} & \square & \boxed{7} \end{array}$$

4.

$$\begin{array}{cccc} & \square & \square & \boxed{8} \\ + & \square & \boxed{3} & \square \\ \hline & \boxed{5} & \boxed{6} & \boxed{7} \end{array}$$

5.

$$\begin{array}{cccc} & \square & \square & \square \\ + & \square & \square & \boxed{6} \\ \hline & \boxed{9} & \boxed{2} & \boxed{7} \end{array}$$

Note that each of the last two questions has multiple solutions.

Variations

Every example in sum is equivalent to two examples in difference. It would be interesting to see if students can make the connection among the solutions of

$$\begin{array}{rccc} & \boxed{1} & \square & \boxed{3} \\ + & \square & \boxed{8} & \square \\ \hline & \boxed{4} & \square & \boxed{9} \end{array} \qquad \begin{array}{rccc} & \boxed{4} & \square & \boxed{9} \\ - & \boxed{1} & \square & \boxed{3} \\ \hline & \square & \boxed{8} & \square \end{array} \quad \text{and} \quad \begin{array}{rccc} & \boxed{4} & \square & \boxed{9} \\ - & \square & \boxed{8} & \square \\ \hline & \boxed{1} & \square & \boxed{3} \end{array}$$

Problem 2: Fill in the digits 0 to 9 (using every digit once) so that

$$\begin{array}{cccc} & \square & \square & \square \\ + & \square & \square & \square \\ \hline \square & \square & \square & \square \end{array}$$

Of course, we would require the sum to be greater than 1000. Otherwise, we are back to the original problem. The first exercise is to show that every sum has the form

$$\begin{array}{cccc} & \square & \square & \square \\ + & \square & \square & \square \\ \hline \boxed{1} & \square & \square & \square \end{array}$$

Then we can assign some digits and construct similar examples:

$$\begin{array}{cccc} & \boxed{4} & \square & \boxed{5} \\ + & \square & \boxed{7} & \square \\ \hline \boxed{1} & \square & \square & \boxed{8} \end{array}$$

We can also consider other patterns:

Problem 3: Fill in the digits 1 to 9 (using every digit once) so that

$$\begin{array}{ccc} & \square & \square \\ & \square & \square \\ + & \square & \square \\ \hline \square & \square & \square \end{array} \qquad \begin{array}{cccc} & & & \square \\ & & \square & \square \\ + & \square & \square & \square \\ \hline & \square & \square & \square \end{array}$$

Problem 4: Fill in the digits 0 to 9 (using every digit once) so that

$$\begin{array}{cccc} & & & \square \\ & & \square & \square \\ + & \square & \square & \square \\ \hline \square & \square & \square & \square \end{array}$$

Another variation is to ask the students to identify which of the following exercises have solutions:

$$\begin{array}{r} \square\ \square\ \square \\ +\ \square\ \square\ \square \\ \hline \boxed{4}\ \boxed{5}\ \boxed{6} \end{array} \qquad \begin{array}{r} \square\ \square\ \square \\ +\ \square\ \square\ \square \\ \hline \boxed{4}\ \boxed{5}\ \boxed{9} \end{array} \qquad \begin{array}{r} \square\ \square\ \square \\ +\ \square\ \square\ \square \\ \hline \boxed{5}\ \boxed{4}\ \boxed{8} \end{array}$$

$$\begin{array}{r} \square\ \square\ \square \\ +\ \square\ \square\ \square \\ \hline \boxed{5}\ \boxed{4}\ \boxed{9} \end{array} \qquad \begin{array}{r} \square\ \square\ \square \\ +\ \square\ \square\ \square \\ \hline \boxed{5}\ \boxed{6}\ \boxed{7} \end{array} \qquad \begin{array}{r} \square\ \square\ \square \\ +\ \square\ \square\ \square \\ \hline \boxed{5}\ \boxed{6}\ \boxed{8} \end{array}$$

$$\begin{array}{r} \square\ \square\ \square \\ +\ \square\ \square\ \square \\ \hline \boxed{6}\ \boxed{4}\ \boxed{8} \end{array} \qquad \begin{array}{r} \square\ \square\ \square \\ +\ \square\ \square\ \square \\ \hline \boxed{7}\ \boxed{8}\ \boxed{3} \end{array} \qquad \begin{array}{r} \square\ \square\ \square \\ +\ \square\ \square\ \square \\ \hline \boxed{7}\ \boxed{8}\ \boxed{4} \end{array}$$

See if any student can recognize that for the following to be solvable,

$$\begin{array}{r} \square\ \square\ \square \\ +\ \square\ \square\ \square \\ \hline \boxed{z_1}\ \boxed{z_2}\ \boxed{z_3} \end{array}$$

we need to have $\boxed{z_1 + z_2 + z_3 = 18}$.

For most students, we don't expect them to be able to prove this. We include the proof here for completeness.

Proof: Suppose

$$\begin{array}{r} \boxed{x_1}\ \boxed{x_2}\ \boxed{x_3} \\ +\ \boxed{y_1}\ \boxed{y_2}\ \boxed{y_3} \\ \hline \boxed{z_1}\ \boxed{z_2}\ \boxed{z_3} \end{array} .$$

Hence,

$$\begin{array}{ll} z_1 + z_2 + z_3 = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) & \text{if there is no carry over} \\ z_1 + z_2 + z_3 = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) - 9 & \text{if there is one carry over} \\ z_1 + z_2 + z_3 = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) - 18 & \text{if there are two carry over} \end{array}$$

$$(\text{Since } (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) + (z_1 + z_2 + z_3) = 1 + 2 + \cdots + 9 = 45)$$

$$\Rightarrow \begin{array}{ll} z_1 + z_2 + z_3 = 45 - (z_1 + z_2 + z_3) & \text{if there is no carry over} \\ z_1 + z_2 + z_3 = 45 - (z_1 + z_2 + z_3) - 9 & \text{if there is one carry over} \\ z_1 + z_2 + z_3 = 45 - (z_1 + z_2 + z_3) - 18 & \text{if there are two carry over} \end{array}$$

$$\Rightarrow \begin{array}{ll} 2\,(z_1 + z_2 + z_3) = 45 & \text{if there is no carry over} \\ 2\,(z_1 + z_2 + z_3) = 36 & \text{if there is one carry over} \\ 2\,(z_1 + z_2 + z_3) = 27 & \text{if there are two carry over} \end{array}$$

Since $z_1 + z_2 + z_3$ is an integer, we have $\boxed{z_1 + z_2 + z_3 = 18}$ and there is one carry over in the sum. Similar argument can be apply to the sums in other patterns.

Note that $\boxed{z_1 + z_2 + z_3 = 18}$ is a necessary but **not sufficient** condition. Discussion on the sufficiency is much more complicated.

Another variation is to replace addition by product.

Problem 5: Fill in the digits 1 to 9 (using every digit once) so that

a)

$$\begin{array}{r} \square\,\square\,\square\,\square \\ \times \qquad\qquad\quad \square \\ \hline \square\,\square\,\square\,\square \end{array}$$

or

b)

$$\begin{array}{r} \square\,\square\,\square \\ \times \qquad\quad \square\,\square \\ \hline \square\,\square\,\square\,\square \end{array}$$

Problem 6: Fill in the digits 0 to 9 (using every digit once) so that

a)

$$\begin{array}{r} \square\,\square\,\square\,\square \\ \times \qquad\qquad\quad \square \\ \hline \square\,\square\,\square\,\square\,\square \end{array}$$

or

b)

$$\begin{array}{r} \square\,\square\,\square \\ \times \qquad\quad \square\,\square \\ \hline \square\,\square\,\square\,\square\,\square \end{array}$$

Similarly, for every example in multiplication, we get two in division.