Problem Given a finite set of complex numbers, determine the smallest regions with special geometrical shape containing them.

1 Convex hull

Let $S$ be a finite set of complex numbers. Here we use an idea by Nam-Kiu Tsing to determine the convex hull of $S$ as follows.

First determine the extrema of the set $\{(z + \bar{z})/2 : z \in S\}$ and $\{(z - \bar{z})/(2i) : z \in S\}$, we get a rectangle touching $k$ vertices of $\text{conv} S$ with $k$. Assume $k > 2$ so that not all points are collinear.

Let $\gamma = (v_1 + \cdots + v_k)/k$ be the mean of the $k$ vertices $v_1, \ldots, v_k$. Replace $S$ by $S - \gamma$. After this replacement, we may assume that $\text{conv} S$ contains a $k$-side convex polygon which contains the origin.

Suppose the complex numbers $v_1, \ldots, v_k$ are $k$ vertices of $\text{conv} S$ that have been identified such that

$$0 \leq \arg(v_1) \leq \cdots \leq \arg(v_k) \leq 2\pi.$$ 

Set $t_{k+1} = t_1$. For $j = 1, \ldots, k$, either

(a) the line segment joining $v_j$ and $v_{j+1}$ is on the boundary $\text{conv} S$, or
(b) the line joining $v_j$ and $v_{j+1}$ separates the origin and a vertex $v_{j'}$.

If (b) does not happen, then we are done. If (b) does happen, we can add $v_{j'}$ to the list of vertices, and do the classification of (a) and (b) again. If $\text{conv} S$ has $m$ vertices, this algorithm will stop after $m - k$ steps.

Note: To check whether (a) or (b) holds, we may let $\theta \in [0, 2\pi)$ such that the line $L$ passing through $e^{i\theta}v_j$ and $e^{i\theta}v_{j+1}$ is a right supporting line of $\text{conv} \{e^{i\theta}v_r : 1 \leq r \leq k\}$. Suppose

$$e^{i\theta}z + e^{-i\theta}\bar{z} \leq e^{i\theta}v_j + e^{-i\theta}\bar{v}_j$$

for all $z \in S$. Then condition (a) holds. Otherwise, condition (b) holds, and we can find $v_{j'}$ among those $z \in S$ such that $e^{i\theta}z$ has the maximum real part.

2 Smallest rectangle

Let $S$ be a finite set of complex numbers. We want to determine the smallest rectangle containing $S$. Taking $P = \text{conv} S$, we reduce the problem to the following.

Problem Determine the smallest rectangle containing a given $n$-side convex polygon $P$.

We can determine the rectangle in finite steps as shown by the following.
Theorem For each side of the polygon $P$, construct a support line $L$ of the polygon. Then there is a unique smallest rectangle with one of the side lying on $L$. There are at most $n$ rectangles constructed in this way. The one with the minimum area is the desired one.

Proof. We claim that if $R$ is a rectangle containing $P$ and none of the four sides of $R$ contains an edge of $P$, then $R$ is not optimal.

Note that if the hypothesis of the claim is true, then there are at most four vertices of $P$ on the boundary of $R$. Moreover, none of these four vertices of $P$ can be a vertex of $R$; otherwise, this vertex and one of the other vertices will be a boundary edge of $P$ lying on the boundary of $R$.

Now, we may rotate $C$ so that the sides of $R$ are $A, B, C, D$ so that $A$ and $C$ are horizontal with $A$ above $C$, and that $B$ and $D$ are vertical with $B$ on the right of $D$. Suppose the vertices $a, b, c, d$ of $P$ lies on the sides $A, B, C, D$ or $R$. Let the line segment joining $a$ and $c$ be $L$, and let the line segment joining $b$ and $d$ be $W$. We may translate $C$ so that the intersection of $L$ and $W$ is the origin. Let $\theta$ be the angle between $L$ and the $y$ axis, and $\phi$ be the angle between $W$ and the $x$ axis. Then the area of $R$ is $lw \cos \theta \cos \phi$, where $l$ and $w$ are the lengths of $L$ and $W$, respectively. If one can rotate $C$ such that both $\theta$ and $\phi$ increase, then we can get a smaller rectangle containing $P$. Otherwise, both $L$ and $W$ must lie in (1) the first and third quadrants, or (2) the second and fourth quadrants. WOLOG, assume (1) holds. If we rotate $C$ by an angle $t$ in the clockwise direction, then the new angles between $L$ and $W$ to the vertical and horizontal axis will be $\cos(\theta + t)$ and $\cos(\phi - t)$, respectively. For sufficiently small $t$, we can find a new rectangle $R(t)$ containing $P$ with area $f(t) = lw \cos(\theta + t) \cos(\phi - t)$. Since

$$f'(t) = -lw(\sin(\theta + t) \cos(\phi - t) - \cos(\theta + t) \sin(\phi - t)) = -lw \sin(\theta + \phi) < 0,$$

we can rotate $C$ and obtain a new rectangle containing $P$ with sides parallel to the axises and a smaller area. Thus, our claim is proved and the result follows.

$\square$

3 Further questions

(a) Determine the smallest ellipse containing $S$.

Every 5 vertices of $\operatorname{conv} S$ determine an ellipse. Can we do better?

(b) How about extending the result to $\mathbb{R}^3$ or higher dimensions?

(c) How about using a (regular) $n$-side polygon to contain a given convex polygon (convex set)?