Problem: Runners at a marathon race are assigned consecutive numbers starting at 1. One of the entrants with a mathematical bent noticed that the sum of the numbers less than his own was equal to the sum of the numbers greater. If there were more than 10 runners but less than 100, what number was he and how many runners were there in the race?

Solution: We consider a more general question: Suppose there were N runners and the runner in question has number K. So we need to solve the equation

$$\frac{N(N+1)}{2} = 2\left(\frac{(K-1)K}{2}\right) + K \quad \Leftrightarrow \quad N(N+1) = 2K^2$$

Since N and $N+1$ are relatively prime, we have either

$$1) \quad \begin{cases} N &= 2q^2 \\ N+1 &= p^2 \end{cases} \qquad \text{or} \qquad 2) \quad \begin{cases} N &= p^2 \\ N+1 &= 2q^2 \end{cases},$$

which give the Pell equations:

$$3) \quad p^2 - 2q^2 = 1 \qquad \text{or} \qquad 4) \quad p^2 - 2q^2 = -1$$

$(p,q) = (3,2)$ is a solution to 3) and the general solutions for 3) are

$$(p,q) = \left(\frac{(3+2\sqrt{2})^n + (3+2\sqrt{2})^n}{2}, \frac{(3+2\sqrt{2})^n - (3+2\sqrt{2})^n}{2\sqrt{2}}\right)$$

for $n = 1, 2, 3, 4, \cdots$.

Similarly, $(p,q) = (1,1)$ is a solution to 4) and the general solutions for 4) are

$$(p,q) = \left(\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}, \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}\right)$$

for $n = 1, 3, 5, \cdots$.

From this, we can find the solutions to the original problem:

$$N = 2\left(\frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{2\sqrt{2}}\right)^2 \qquad \text{for } n = 1, 2, 3, 4, \cdots \qquad \text{or}$$

$$N = \left(\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}\right)^2 \qquad \text{for } n = 3, 5, \cdots \ (\ N > 1\ )$$

and $K = \sqrt{\dfrac{N(N+1)}{2}}$.

K	N	K	N
6	8	271669860	384199200
35	49	1583407981	2239277041
204	288	9228778026	13051463048
1189	1681	53789260175	76069501249
6930	9800	313506783024	443365544448
40391	57121	1827251437969	2584123765441
235416	332928	10650001844790	15061377048200
1372105	1940449	62072759630771	87784138523761
7997214	11309768	361786555939836	511643454094368
46611179	65918161	2108646576008245	2982076586042449

Variation: Runners at a marathon race are assigned consecutive numbers starting at 1. One of the entrants with a mathematical bent noticed that the sum of the numbers less than **or equal to** his own was equal to the sum of the numbers greater. If there were more than 10 runners but less than 100, what number was he and how many runners were there in the race?

Solution: We consider a more general question: Suppose there were N runners and the runner in question has number K. So we need to solve the equation

$$\frac{N(N+1)}{2} = 2\left(\frac{(K+1)K}{2}\right) \quad \Leftrightarrow \quad N(N+1) = 2K(K+1)$$

$$\Leftrightarrow \quad 4N(N+1) - 2(4K(K+1)) = 0 \quad \Leftrightarrow \quad (2N+1)^2 - 2(2K+1)^2 = -1$$

Let $p = 2N+1$ and $q = 2K+1$. We have the Pell equation:

$$p^2 - 2q^2 = -1$$

which have the general solutions

$$(p,q) = \left(\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}, \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}\right)$$

for $n = 3, 5, \cdots$. $(q > 1)$

From this, we can find the solutions to the original problem:

$$N = \frac{1}{2}\left(\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2} - 1\right) \qquad \text{and } K = \frac{1}{2}\left(\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} - 1\right)$$

for $n = 3, 5, \cdots$.

K	N	K	N
2	3	112529340	159140519
14	20	655869060	927538920
84	119	3822685022	5406093003
492	696	22280241074	31509019100
2870	4059	129858761424	183648021599
16730	23660	756872327472	1070379110496
97512	137903	4411375203410	6238626641379
568344	803760	25711378892990	36361380737780
3312554	4684659	149856898154532	211929657785303
19306982	27304196	873430010034204	1235216565974040