Part 5 Functions and Matrices

We study functions on matrices related to the Löwner (positive semidefinite) ordering on positive semidefinite matrices that $A \geq B$ means $A - B$ is positive semi-definite. Note that every function $f : \mathbb{C} \to \mathbb{C}$ can be extended to diagonalizable matrices using the spectral decomposition

$$A = S^{-1} \text{diag}(a_1, \ldots, a_n)S \mapsto S^{-1} \text{diag}(f(a_1), \ldots, f(a_n))S.$$ 

More generally, if $A$ has minimal polynomial $(z - a_1)^{r_1} \cdots (z - a_k)^{r_k}$, and $f$ has derivatives at a neighborhoods of $a_j$ up to order $r_j - 1$, then $f(A)$ can be defined, and has the same value as $p(A)$ for a polynomial interpolating the values and derivatives of $f(a_j)$ up to order $r_j - 1$ for $j = 1, \ldots, k$.

1 Positive and completely positive linear maps

**Definition 1.1** A function $\Phi : M_n \to M_m$ is a **positive linear map** if $\Phi(A) \geq \Phi(B)$ whenever $A \geq B$. It is unital if $\Phi(I) = I$. It **$k$-positive** if

$$\Phi \otimes I_k(A_{ij})_{1 \leq i,j \leq k} = (\Phi(A_{ij}))_{1 \leq i,j \leq k}$$

is positive whenever $(A_{ij}) \in M_k(M_n)$ is positive. If $\Phi$ is $k$-positive for all $k = 1, 2, \ldots$, then $\Phi$ is **completely positive**.

**Lemma 1.2** Let $A, B \in H_n$.

(a) Then $A \leq B$ if and only if $X^*AX \leq X^*BX$ for any nonzero $n \times m$ matrix $X$.

(b) Suppose $0 \leq A, B$ and $B$ is invertible. Then $A \leq B$ is and only if $\|A^{1/2}B^{-1/2}\| \leq 1$.

**Lemma 1.3** A matrix $C = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in H_n$ is positive semi-definite if and only if $B - X^*A^{-1}X$ is positive semi-definite.

**Lemma 1.4** Suppose $\Phi$ is a positive linear map. If $A, B, C \in H_n$ are such that $\begin{pmatrix} A & C \\ C & B \end{pmatrix} \geq 0$, then $\begin{pmatrix} \Phi(A) & \Phi(C) \\ \Phi(C) & \Phi(B) \end{pmatrix} \geq 0$.

**Corollary 1.5** Suppose $\Phi$ is a positive linear map. If $A \geq 0$, then

$$\Phi(A^2) \geq \phi(A)^2 \quad \text{and} \quad \Phi(A^{-1}) \geq \Phi(A)^{-1}.$$
Remark 1.6 For a fixed $n \times m$ matrix $X$ such that $X^*X = I_m$, the map defined by $\Phi(A) = X^*AX$ is unital and completely positive.

Suppose $P_1, \ldots, P_k \in M_n$ are orthogonal projections such that $P_1 + \cdots + P_k = I$. The map $A \mapsto P_1AP_1 + \cdots + P_kAP_k$ is unital and completely positive.

Theorem 1.7 Let $\Phi : M_n \to M_m$. The following are equivalent.

(a) $\Phi$ is completely positive.

(b) $(\Phi(E_{ij}))_{1 \leq i, j \leq n}$ is positive semidefinite.

(c) There are $n \times m$ matrices $X_1, \ldots, X_k$ such that

$$\Phi(A) = \sum_{j=1}^{k} X_j^*AX_j.$$

2 Operator monotone and operator convex functions

Definition 2.1 Let $f : \mathbb{R} \to \mathbb{R}$.

(a) The function $f$ is matrix monotone of order $n$ if for all $A, B \in H_n$ and all $t \in [0, 1],

$$f(A) \leq f(B) \text{ whenever } A \leq B.$$ If this is true for all orders, then $f$ is operator monotone.

(b) The function $f$ is matrix convex of order $n$ if for all $A, B \in H_n$ and all $t \in [0, 1],

$$f((1-t)A + tB) \leq (1-t)f(A) + tf(B).$$ If this is true for all orders, then $f$ is operator monotone.

If the condition on $f$ holds for $t = 1/2$, then $f$ is mid-point matrix/operator convex. For continuous functions, mid-point convex implies matrix/operator convex.

(c) The function $f$ is matrix concave of order $n$ if $-f$ is matrix convex of order $n$.

Example 2.2 Here are some basic examples.

1. The function $f(t) = c + dt$ is operator monotone and operator convex if $d \geq 0$ and $c \in \mathbb{R}$. It is operator convex for all $c, d \in \mathbb{R}$.

2. The function $f(t) = t^2$ is operator convex, but not operator monotone.

3. The function $f(t) = t^3$ is not operator convex, and not operator monotone.

4. The function $f(t) = |t|$ is not operator convex.
Theorem 2.3 (a) \( f(t) = t^{-1} \) is operator monotone on \((0, \infty)\).
(b) If \( r \in [0, 1] \), then \( f(t) = t^r \) is operator monotone on \([0, \infty)\).

Theorem 2.4 If \( f \) is operator monotone on \([0, \infty)\), then there are \( a, b \in \mathbb{R} \) with \( b \geq 0 \) and a positive measure \( \mu \) on \([0, \infty)\) such that
\[
 f(t) = a + bt + \int_0^\infty \frac{st}{s+t} d\mu(s).
\]
If \( f \) is operator convex on \([0, \infty)\), then there are \( a, b, c \in \mathbb{R} \) with \( c \geq 0 \) and a positive measure \( \mu \) on \([0, \infty)\) such that
\[
 f(t) = a + bt + ct^2 + \int_0^\infty \frac{s^2 t}{s+t} d\mu(s).
\]

Theorem 2.5 Let \( A, B \in \mathcal{H}_n \) be positive semidefinite, and \( \| \cdot \| \) be a UI norm. If \( f(t) \) is a nonnegative operator monotone function \( f(t) \) on \([0, \infty)\), then
\[
 \| f(A + B) \| \leq \| f(A) + f(B) \|.
\]
Furthermore, if \( f^{-1} \) exists such that \( f(0) = 0 \) and \( f(\infty) = \infty \), then
\[
 \| f^{-1}(A + B) \| \geq \| f^{-1}(A) + f^{-1}(B) \|.
\]

Corollary 2.6 Let \( \| \cdot \| \) be a UI norm on \( M_n \). If \( A, B \in \mathcal{H}_n \) are positive semidefinite and \( r \in (0, 1] \) then
\[
 \|(A + B)^r\| \leq \|A^r + B^r\| \quad \|(A + B)^{1/r}\| \geq \|A^{1/r} + B^{1/r}\|,
\]
and
\[
 \| \log(A + B + I) \| \leq \| \log(A + I) + \log(B + I) \| \quad \text{and} \quad \| e^A + e^B \| \leq \| e^{A+B} + I \|.
\]

It is known that a function \( g(t) \) on \([0, \infty)\) with \( g(0) = 0 \) is operator convex if and only if \( g(t)/t \) is operator monotone on \((0, \infty)\). We have the following corollary.

Corollary 2.7 Let \( g(t) \) be a nonnegative operator convex function on \([0, \infty)\) with \( g(0) = 0 \). If \( \| \cdot \| \) is a UI norm on \( M_n \), then
\[
 \| g(A + B) \| \geq \| g(A) + g(B) \|
\]
for any positive semidefinite \( A, B \in \mathcal{H}_n \).

Theorem 2.8 Let \( f(t) \) be a nonnegative operator monotone function on \([0, \infty)\) and \( \| \cdot \| \) be a UI norm on \( M_n \). Let \( A \in M_n \).

(a) If \( \| \text{diag}(1, 0, \ldots, 0) \| = 1 \), then
\[
 f(\|A\|) \leq f(|A|)\|.
\]
(b) If \( \| I \| = 1 \), then
\[
 f(\|A\|) \geq f(|A|)\|.
\]
3 Extremal representation techniques

Proposition 3.1 Let $A \in H_n$ be positive definite, and $B \in M_n$. Then

$$B^* A^{-1} B = \min \left\{ C : \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \right\}.$$  

In particular,

$$A^{-1} = \min \left\{ C : \begin{pmatrix} A & I \\ I & C \end{pmatrix} \geq 0 \right\}.$$  

Corollary 3.2 If $A \succeq B$ then $A^{-1} \leq B^{-1}$.

Proposition 3.3 Note that

$$A^{1/2} = \max \left\{ C : \begin{pmatrix} A & C \\ C & I \end{pmatrix} \geq 0 \right\}.$$  

Thus, $A \succeq B$ implies that $A^{1/2} \succeq B^{1/2}$.

Proposition 3.4 Suppose $A = (A_{ij})_{1 \leq i,j \leq 2} \in H_n$ is positive definite. Then the Schur-complement of $A$ with respect to $A_{11}$ given by $S(A) = A_{22} - A_{21} A_{11}^{-1} A_{12}$ satisfies

$$S(A) = \max \{ C : A \succeq 0 \oplus C \text{ with } C \in H_{n-k} \}$$

and

$$S(A) = \min \{ [Z|I_{n-k}]A[Z|I_{n-k}]^* : Z \text{ is } (n-k) \times k \}.$$  

Corollary 3.5 Suppose $f$ is operator monotone on $[0, \infty)$ and $f(0) \geq 0$. Then for any positive linear map from $M_n$ to $M_{n-k}$, and any positive definite matrix $A$,

$$f(\Phi(A)) \geq \Phi(f(A)) \geq S(f(A)) \geq f(S(A)).$$  

In particular,

$$[\Phi(A^p)]^{1/p} \geq \Phi(A) \geq S(A) \geq [S(A^p)]^{1/p} \quad \text{for } p \geq 1,$$

and

$$[\Phi(A^p)]^{1/p} \leq \Phi(A) \leq S(A) \leq [S(A^p)]^{1/p} \quad \text{for } p \leq -1.$$  

Corollary 3.6 Suppose $A \in H_n$ is positive definite and $B \in M_n$. Then for any $1 \leq i_1 < \cdots < i_m \leq k$ and $1 \leq j_1 < \cdots < j_m \leq n$. If $i_m + j_m \leq m + k$, then

$$\prod_{s=1}^m \lambda_{i_s+j_s-s}(S(BAB^*)) \leq \prod_{s=1}^m \lambda_{i_s}(S(BB^*))^{n-j_s}(A).$$
4 Current research

**Theorem 4.1** Suppose \( f : [0, \infty) \to [0, \infty) \) is concave. If \( A, B \in H_n \) are positive semidefinite, then
\[
\|f(A + B)\| \leq \|f(A) + f(B)\|
\]
for any UI norm.

**Problem 4.2** Suppose \( f : [0, \infty) \to [0, \infty) \) is concave. If \( A, B \in M_n \) are normal. Is it true that
\[
\|f(|A + B|)\| \leq \|f(|A|) + f(|B|)\|
\]

5 Exercises

1. Suppose \( \Phi \) is a positive linear map. Show that
\[
\|\Phi\| = \max\{\|\Phi(A)\| : s_1(A) \leq 1\} = \|\Phi(I)\|.
\]

2. Show that \( \|\Phi\| \) is attained at a unitary, and show that
\[
\begin{pmatrix}
\Phi(I) & \Phi(U) \\
\Phi(U)^* & \Phi(I)
\end{pmatrix} \succeq 0
\]

\[
\text{to conclude that } \|\Phi(U)\| \leq \|\Phi(I)\| \text{ for any unitary } U.
\]

3. Suppose \( \Phi \) is a positive linear map such that \( \Phi(I) \leq I \). Show that \( \|\Phi(A)\| \leq \|A\| \) for any UI norm \( \|\cdot\| \).

4. Show that the scalar function \( f(x) = |x| \) on \( M_1 \) is 2-positive but not 3-positive.

5. Show that the scalar function \( f(x) = |x| \) is not operator convex.