Part 4 Numerical ranges and quantum computing

The **numerical range** and the **numerical radius** of $A \in M_n$ are defined as

$$W(A) = \{x^*Ax : x \in \mathbf{C}^n,\ x^*x = 1\} \qquad \text{and} \qquad r(A) = \max\{|\mu| : \mu \in W(A)\}.$$

These concepts are useful in the study of matrices. There are many generalizations motivated by applications. We discussed some basic properties and selected generalizations useful in quantum computing.

1 The classical numerical range

Proposition 1.1 *Let $A \in M_n$.*

1. $W(U^*AU) = W(A)$ *for any unitary* $U \in M_n$.
2. $W(A + cI) = W(A) + c$ *for any* $c \in \mathbf{C}$.
3. $W(cA) = cW(A)$ *for any* $c \in \mathbf{C}$.
4. $\sigma(A) \subseteq W(A)$.
5. $W(A + B) \subseteq W(A) + W(B)$ *for any* $B \in M_n$.
6. $W(A \oplus B) = \mathbf{conv}\,[W(A) \cup W(B)]$ *for any* $B \in M_m$.
7. $W(A) = \mathbf{conv}\,\{a_1, \ldots, a_n\}$ *if A is normal with eigenvalues* $a_1, \ldots, a_n$.

Theorem 1.2 *For any $A \in M_n$, $W(A)$ is a compact convex set in $\mathbf{C}$. If $A \in M_2$ then $W(A)$ is an elliptical disk with the eigenvalues a_1, a_2 of A as foci and $\gamma = \sqrt{\mathrm{tr}\,(A^*A) - |a_1|^2 - |a_2|^2}$ as minor axis.*

Theorem 1.3 *Let $A \in M_n$.*

1. $W(A) = \{\mu\}$ *if and only if* $A = \mu I$.
2. $W(A) \subseteq a\mathbf{R} + b$ *if and only if* $A = aH + bI$ *with* $H = H^*$.
3. *A is unitary if and only if A is invertible such that both $W(A)$ and $W(A^{-1})$ are subsets of the closed unit disks.*

Theorem 1.4 *Let $A \in M_n$. Then $\mathrm{Re}(W(A)) = W((A + A^*)/2)$. Consequently,*

$$W(A) = \{\mu \in \mathbf{C} : e^{it}\mu + e^{-it}\bar{\mu} \le \lambda_1(e^{it}A + e^{-it}A^*), t \in [0, 2\pi)\}.$$

Theorem 1.5 *Let $A \in M_3$ be a unitarily reducible matrix or $A \in M_2$. Then $B \in M_n$ satisfies $W(B) \subseteq W(A)$ if and only if $B = X^*(A \otimes I_m)X$ for some matrix X of appropriate size such that $X^*X = I_n$.*

Theorem 1.6 *Let A and B be square matrices. Define the function Φ from* span $\{I, A, A^*\}$ *to* span $\{I, B, B^*\}$ *by $\Phi(aI + bA + cA^*) = aI + bB + cB^*$.*

(a) *Then $W(B) \subseteq W(A)$ if and only if Φ is a positive linear map.*

(b) *The matrix B is a compression of $A \otimes I$ if and only if Φ is a completely positive linear map.*

Theorem 1.7 *Let $A \in M_n$. Then*

$$\rho(A) \leq r(A) \leq \|A\| \leq 2r(A)$$

and

$$r(A^k) \leq r(A)^k, \qquad k = 1, 2, \ldots.$$

(a) *The equality $\rho(A) = r(A)$ holds if and only if A is unitarily similar to a matrix of the form $[\mu] \oplus A_2$ such that $|\mu| = r(A)$.*

(b) *The equality $\rho(A) = \|A\|$ holds if and only if $r(A) = \|A\|$. This happens if and only if A is unitarily similar to a matrix of the form $[\mu] \oplus A_2$ such that $|\mu| = \|A\|$.*

(c) *The equality $\|A\| = 2r(A)$ holds if and only if $A/r(A)$ is unitarily similar to a matrix of the form $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \oplus A_2$ with $r(A_2) \leq 1$.*

Theorem 1.8 *Let $A, B \in M_n$. Then*

$$r(AB) \leq 4r(A)r(B).$$

If $AB = BA$, then

$$r(AB) \leq 2r(A)r(B).$$

Problem 1.9 *Determine the best (smallest) constant γ such that $r(AB) \leq \gamma r(A)\|B\|$ for A and B such that $AB = BA$.*

Problem 1.10 *Determine the best (smallest) constant γ such that*

$$\|p(A)\| \leq \gamma \max\{|p(\mu)| : \mu \in W(A)\}$$

for any complex polynomial $p(z)$.

2 The higher rank numerical range

In connection to quantum error correction, see the appendix, researchers consider the **rank k-numerical range** of $A \in M_n$ defined by

$$\Lambda_k(A) = \{\mu \in \mathbf{C} : \text{ there is } P \in \mathcal{P}_k \text{ such that } PAP = \mu P\},$$

where $\mathcal{P}_k$ is the set of rank k-orthogonal projections in M_n.

Theorem 2.1 *Let $A \in M_n$ and $1 \le k \le n$.*

1. *For any $a, b \in \mathbf{C}$, $\Lambda_k(aA + bI) = a\Lambda_k(A) + b$.*
2. *For any unitary $U \in M_n$, $\Lambda_k(U^*AU) = \Lambda_k(A)$.*
3. *If $B \in M_r$ is a compression of A with $r \ge k$, then $\Lambda_k(B) \subseteq \Lambda_k(A)$.*
4. *Suppose $n < 2k$. The set $\Lambda_k(A)$ has at most one element.*

Theorem 2.2 *Let $w = e^{i2\pi/3}$ and*

$$B = I_{k-1} \oplus wI_{k-1} \oplus w^2 I_{k-1}.$$

If $n \le 3k-3$, then for any $(3k-3) \times n$ matrix X satisfying $X^*X = I_n$, $\Lambda_k(X^*BX) = \emptyset$. If $n \ge 3k-2$ then $\Lambda_k(A)$ is non-empty for any $A \in M_n$.

Theorem 2.3 *Let $A \in M_n$. Then $\Lambda_k(A) = \Omega_k(A)$, where*

$$\Omega_k(A) = \bigcap_{\xi \in [0,2\pi)} \left\{\mu \in \mathbf{C} : e^{i\xi}\mu + e^{-i\xi}\bar{\mu} \le \lambda_k(e^{i\xi}A + e^{-i\xi}A^*)\right\}.$$

In particular, if $A \in M_n$ is a normal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$, then

$$\Lambda_k(A) = \bigcap_{1 \le j_1 < \cdots < j_{n-k+1} \le n} \mathbf{conv}\,\{\lambda_{j_1}, \dots, \lambda_{j_{n-k+1}}\}.$$

Corollary 2.4 *For any $A \in M_n$ and $1 \le k \le n$, $\Lambda_k(A)$ is convex.*

3 The joint higher rank numerical range

Definition 3.1 *Let* $A_1, \ldots, A_m \in M_n$. *The* **joint rank-**k **numerical range** *of* $\mathbf{A} = (A_1, \ldots, A_m)$ *is defined by*

$$\Lambda_k(\mathbf{A}) = \{(a_1, \ldots, a_m) : \text{ there is } P \in \mathcal{P}_k \text{ such that } PA_jP = a_jP, \ j = 1, \ldots, m\},$$

where $\mathcal{P}_k$ *is the set of rank* k *orthogonal projections in* M_n.

Remark 3.2 *If* $A_j = H_j + iG_j$ *with* $H_j = H_j^*$ *and* $G_j = G_j^*$, *then* $\Lambda_k(A_1, \ldots, A_m) \subseteq \mathbf{C}^{1\times m}$ *can be identified as* $\Lambda_k(H_1, G_1, \ldots, H_m, G_m) \subseteq \mathbf{R}^{1\times 2m}$. *So, we may focus on the joint rank* k*-numerical range of Hermitian matrices.*

Proposition 3.3 *Suppose* $A_1, \ldots, A_m \in H_n$. *Let* $T = (t_{ij}) \in M_m(\mathbf{R})$ *and* $(c_1, \ldots, c_m)$ *be a real vectors. If* $B_j = c_jI + \sum_{j=1}^m t_{ij}A_i$, *then*

$$\Lambda_k(B_1, \ldots, B_m) = \{(c_1, \ldots, c_m) + (a_1, \ldots, a_m)T : (a_1, \ldots, a_m) \in \Lambda_k(A_1, \ldots, A_m)\}.$$

Theorem 3.4 *Let* $A_1, \ldots, A_m \in H_n$. *Then* $W(A_1, \ldots, A_m)$ *is convex if*

(a) span $\{I, A_1, \ldots, A_m\}$ *has dimension at most 3, or*

(b) $n \geq 3$ *and* span $\{I, A_1, \ldots, A_m\}$ *has dimension at most 4.*

Example 3.5 *Let*

$$B_1 = I_2 \oplus 0_{n-2}, \ B_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus 0_{n-2}, \ B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0_{n-2}, \ B_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus 0_{n-2}.$$

Then $W(B_1, B_2, B_3, B_4)$ *is not convex.*

Theorem 3.6 *Let* $A_1, A_2, A_3 \in H_n$ *be such that* span $\{I, A_1, A_2, A_3\}$ *has dimension 4. Then there is* A_4 *such that* $W(A_1, A_2, A_3, A_4)$ *is not convex.*

There are many problems on $\Lambda_k(A_1, \ldots, A_m)$ under active research.

Problem 3.7 *Let* $A_1, \ldots, A_m \in H_n$. *For* $k > 1$ *the set* $\Lambda_k(A_1, \ldots, A_m)$ *may be empty. Determine the minimum* n *(in terms of* m *and* k*) so that* $\Lambda_k(A_1, \ldots, A_m)$ *is always non-empty for* $A_1, \ldots, A_m \in H_n$.

Theorem 3.8 *For* $m, k \geq 1$, *let*

$$n(m,k) = \begin{cases} 2 \cdot 3^{\frac{m-1}{2}}(k-1) + 1 & \text{when } m \text{ is odd}, \\ \\ 3^{\frac{m}{2}}(k-1) + 1 & \text{when } m \text{ is even}. \end{cases}$$

Then $\Lambda_k(A_1, \ldots, A_m)$ *is non-empty for all* $A_1, \ldots, A_m \in \mathcal{H}_n$.

Example 3.9 *For $m, k > 1$, let $n = (m+1)(k-1)$. Suppose $A_1 = I_{k-1} \oplus 0_{k-1} \oplus -I_{(m-1)(k-1)}$ and*

$$A_j = I_{j(k-1)} \oplus 0_{(m+1-j)(k-1)}, \quad j = 2, \ldots, m.$$

Then $\Lambda_k(A_1, \ldots, A_m) = \emptyset$.

Proposition 3.10 *Suppose $A_1, \ldots, A_m \in H_n$ are diagonal matrices. If $n > m + 1$, then $\Lambda_2(A_1, \ldots, A_m) \neq \emptyset$.*

Problem 3.11 *Can we extend the above result to general Hermitian matrices $A_1, \ldots, A_m$?*

Theorem 3.12 *Let $\mathbf{A} = (A_1, \ldots, A_m) \in H_n^m$. If $(a_1, \ldots, a_m) \in \Lambda_{\hat{k}}(\mathbf{A})$, where $\hat{k} \geq (m+2)k$ if $k > 1$ and $\hat{k} \geq (m+1)/2$ if $k = 1$. Then $\Lambda_k(A_1, \ldots, A_m)$ is star-shaped with $(a_1, \ldots, a_m)$ as a star-center. Consequently,* **conv** *$\Lambda_{\hat{k}}(A_1, \ldots, A_m)$ is a compact convex subset of $\Lambda_k(\mathbf{A})$.*

Problem 3.13

1. *Determine the minimum n such that $\Lambda_k(A_1, \ldots, A_m)$ is star-shaped for any $A_1, \ldots, A_m \in H_n$.*
2. *Determine the condition on $A_1, \ldots, A_m \in H_n$ so that $\Lambda_k(A_1, \ldots, A_m)$ is convex.*
3. *Determine a "large" convex subset of $\Lambda_k(A_1, \ldots, A_m)$.*

4 The C-numerical range and quantum control

Definition 4.1 *Let $C \in M_n$. The* **C-numerical range** *and the* **C-numerical radius** *of $A \in M_n$ are defined by*

$$W_C(A) = \{\mathrm{tr}\,(CU^*AU) : U \text{ is unitary}\}$$

and

$$r_C(A) = \max\{|\mu| : \mu \in W_C(A)\}.$$

Note that the C-numerical radii are the building blocks for USI norms on M_n.

Theorem 4.2 *Suppose $C = aI + bR$ where R is Hermitian or rank one. Then $W_C(A)$ is convex for any $A \in M_n$.*

Definition 4.3 *A matrix C is a* **block shift operator** *if it is unitarily similar to a block matrix $(C_{ij})_{1 \leq i,j \leq m}$ such that $C_{11}, \ldots, C_{mm}$ are square matrices, and $C_{ij} = 0$ whenever $i \neq j + 1$.*

Theorem 4.4 *Suppose $C = aI + R$ where R is a block shift operator. Then $W_C(A)$ is a circular disk for any $A \in M_n$.*

Problem 4.5 *Characterize matrices $C \in M_n$ such that $W_C(A)$ is convex for all $A \in M_n$.*

Definition 4.6 *Let $C \in M_n$ have eigenvalues $c_1, \ldots, c_n$. Define the* **C-spectral radius** *and* **C-spectral norm** *of $A \in M_n$ by*

$$\rho_C(A) = \max\left\{\left|\sum_{j=1}^{n} c_{i_j}\lambda_j(A)\right| : (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n)\right\},$$

and

$$\|A\|_C = \max\{|tr(CUAV) : U, V \text{ are unitary}\}.$$

Theorem 4.7 *Let $C \in M_n$ have singular values $c_1 \geq \cdots \geq c_n$. Then*

$$\|A\|_C = \sum_{j=1}^{n} c_j s_j(A).$$

Note that the C-spectral norms are the building blocks of UI norms on M_n.

In quantum control, it is important to determine

$$\min\{\|C - U^*BU\| : U \text{ is unitary}\}$$

for two given (nilpotent) matrices C and A arising from some quantum mechanical systems.

Note that

$$\|C - U^*BU\|^2 = \|C\|^2 + \|A\|^2 - 2\mathrm{Re}(\mathrm{tr}\,(CU^*B^*U)).$$

So, the problem reduces to finding

$$r_C(B^*) = \max\{\mathrm{Re}(\mathrm{tr}\,(CU^*B^*U)) : U \text{ is unitary}\}.$$

Problem 4.8 *Determine $r_{C_k}(A_k)$ for*

$$C_k = \begin{pmatrix} 0_{2^k} & 0_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix} \qquad \text{and} \qquad A_k = N_k \oplus N_k,$$

where

$$N_0 = (0), \qquad N_k = \begin{pmatrix} N_{k-1} & 0 \\ I_{2^{k-1}} & N_{k-1}. \end{pmatrix}$$

Here are some conjectured values:

k:	3	4	5	6
$r_{C_k}(A_k)$:	$4(1+\sqrt{3})$	$8(1+\sqrt{3})$	$16(1+\sqrt{3})+4\sqrt{5}$	$32(1+\sqrt{3})+8\sqrt{5}$

Recently, researchers study the local C-numerical range and C-numerical radius with respect to a certain subgroup $\mathcal{S}$ of the unitary group defined by

$$W_{\mathcal{S}(C)}(A) = \{\mathrm{tr}\,(CU^*AU) : U \in \mathcal{S}\}$$

and

$$r_{\mathcal{S}(C)}(A) = \{|\mu| : \mu \in W_{\mathcal{S}(C)}(A)\}.$$

5 Exercises

1. Suppose $\mu \in \sigma(A)$ is a boundary point of $W(A)$. Show that A is unitarily similar to $[\mu] \oplus A_2$.

2. Show that if $\mu \in W(A)$ satisfies $|\mu| = \|A\|$, then A is unitarily similar to $[\mu] \oplus A_2$.

3. Show that if $A \in M_n$ and $W(A)$ is a convex polygon (with interior) with $n-1$ vertices, then A is normal. For each $n \geq 5$, show that there is a non-normal matrix B such that $W(B)$ is a convex polygon with $n-2$ vertices.

4. Let
$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$
Show that
$$W(A_1, A_2, A_3) = \{(a,b,c) : a,b,c \in \mathbf{R}, a^2+b^2+c^2 = 1\}.$$

5. Give a complete description of $\Lambda_2(A)$ for a normal matrix $A \in M_4$ in terms of its eigenvalues.

6. Show that if $A \in M_n$ has rank less than k, then $\Lambda_k(A) = \{0\}$.

7. Suppose $n \geq 2k$. There is $A \in M_n$ such that $\Lambda_k(A)$ is the unit circular disk.

8. Suppose $n \geq 2k+m$. There is $A \in M_n$ such that $\Lambda_k(A)$ is a regular m-side polygons.

9. If $W_C(A)$ is always a circular disk centered at the origin, show that C is a block shift operator.
   If $W_C(A)$ is always a circular disk, can we conclude that $C = aI + R$ for a block shift operator R?

Appendix: Background of quantum error correction

In classical computing, information is stored as binary sequences. A length k sequence is encoded as a length n sequence, and then transmitted through a noisy channel. The received sequence can be correctly decoded provided there are fewer than $r(n,k)$ error.

In quantum computing, information is stored in **quantum bits (qubits)**. Mathematically, a qubit is represented by a 2×2 rank one projection $Q = \frac{1}{2}\begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix}$ with $x^2+y^2+z^2 = 1$. A state of K-qubits $Q_1, \dots, Q_K$ is represented by their tensor products in M_n with $k = 2^K$. Again, a state of K-qubits is encoded as a state of N-qubits, and transmitted through a **quantum channel**, where a quantum channel for states of N-qubits is a **trace preserving completely positive linear map** $\Phi : M_n \to M_n$ with $n = 2^N$. By the result of Choi, there are $T_1, \dots, T_m \in M_n$ with $\sum_{j=1}^m T_j^* T_j = I_n$ such that

$$\Phi(X) = \sum_{j=1}^{m} T_j X T_j^*. \tag{1}$$

In this setting an quantum error correction code is a subspace $\mathbf{V}$ of $\mathbf{C}^n$ such that the compression of Φ on $\mathbf{V}$ is the identity map. By the result of Knill-Laflamme, this happens if and only if there are scalars γ_{ij} with $1 \le i, j \le r$ such that

$$PT_i^* T_j P = \gamma_{ij} P, \quad 1 \le i, j \le m,$$

where $P \in M_n$ is an orthogonal projection of $\mathbf{C}^n$ onto $\mathbf{V}$.

In connection to this, researchers study the joint rank-k numerical range of $(A_1, \dots, A_m)$ to be the set $\Lambda_k(A_1, \dots, A_m)$ of complex vectors $(a_1, \dots, a_m) \in \mathbf{C}^{m \times 1}$ for the existence of an rank-k orthogonal projection $P \in M_n$ such that $PA_jP = a_jP$ for $j = 1, \dots, m$.