Part 3 Norms and norm inequalities

The study of norms has connections to many pure and applied areas. We will focus on approximation problems and norm inequalities in matrix spaces.

1 $S$-invariant norms

Definition 1.1 A norm $\| \cdot \|$ on a vector space $V$ is a function from $V$ to $\mathbb{R}$ such that

(a) $\|v\| \geq 0$ for all $v \in V$, where $\|v\| = 0$ if and only if $v = 0$.

(b) $|\gamma v| = |\gamma|\|v\|$ for all $\gamma \in \mathbb{F}$ and $v \in V$.

(c) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

If the condition “$\|v\| = 0$ if and only if $v = 0$” is relaxed, we have a semi-norm.

Definition 1.2 Let $S$ be a set of operators acting on the vector space $V$. A norm $\| \cdot \|$ on $V$ is $S$-invariant if $\|Av\| = \|v\|$ for all $A \in S$ and $v \in V$.

Example 1.3 Here are several commonly used $S$-invariant norms on vectors or matrices.

1. Absolute norms on $\mathbb{F}^n$, i.e., $\|Dv\| = \|v\|$ for all diagonal unitary / diagonal orthogonal matrices $D$.

2. Permutationally invariant norms on $\mathbb{F}^n$, i.e., $\|Pv\| = \|v\|$ for all permutation matrices $P$.

3. Symmetric norms (also known as symmetric gauge functions) on $\mathbb{R}^n$, i.e, absolute permutationally invariant norms.

4. Unitarily invariant norms on $m \times n$ matrices, i.e., $\|UAV\| = \|A\|$ for all unitary matrices $U$ and $V$.

5. Unitary similarity invariant norms on $M_n$ or $H_n$, i.e., $\|U^*AU\| = \|A\|$ for all unitary matrices $U$.

6. Unitary congruence invariant norms on $M_n$, symmetric matrices, or skew-symmetric matrices, i.e., $\|U^TAU\| = \|A\|$ for all unitary matrices $U$.

Theorem 1.4 Suppose $V$ is an inner product space and $S$ is a group of unitary operators.

(a) For $\gamma \in V$, the $S(\gamma)$-radius defined by

$$r_\gamma(v) = \max\{|\langle v, g(\gamma) \rangle| : g \in S\}$$

is an $S$-invariant semi-norm. It is a norm if and only if $S(\gamma) = \{g(\gamma) : g \in S\}$ spans $V$. 

1
For every $S$-invariant norm $\| \cdot \|$ on $V$, there is a compact subset $E$ of $V$ such that

$$\|v\| = \max \{ r_\gamma(v) : \gamma \in E \}.$$ 

For any $u, v \in V$, $\|u\| \leq \|v\|$ for all $S$-invariant norms if and only if $r_\gamma(u) \leq r_\gamma(v)$ for all $\gamma \in V$ (or a suitable collection of $\gamma$).

**Remark 1.5** The theorem shows that $S(\gamma)$-radii are the building blocks of $S$-norms if $S$ is a group of unitary operators.

### 2 UI norms, USI norms, and UCI norms

**Definition 2.1** Let $c = (c_1, \ldots, c_n)$ be a vector with nonnegative entries arranged in descending order. The $c$-norm (or $c$-spectral norm) on $M_n$ is defined by

$$\|A\|_c = \sum_{j=1}^{n} c_j s_j(A).$$

When $c_1 = \cdots = c_k = 1$ and $c_j = 0$ for $j > k$, have the Ky Fan $k$-norm $F_k(A)$.

**Theorem 2.2** There is a one-one correspondence between symmetric gauge functions on $\mathbb{R}^n$ and UI norms on $M_n$, namely, each UI norm $\| \cdot \|$ corresponds to a unique symmetric gauge function $\Phi$ such that $\|A\| = \Phi(s(A))$.

(a) Every UI norm $\| \cdot \|$ there is a compact set $E$ of nonnegative vectors with entries arranged in descending order such that

$$\|A\| = \max \{ \|A\|_c : c \in E \}.$$ 

(b) Let $A, B \in M_n$. Then $\|A\| \leq \|B\|$ for all UI norms $\| \cdot \|$ if and only if $F_k(A) \leq F_k(B)$ for all $k = 1, \ldots, n$.

**Theorem 2.3** There is a one-one correspondence between permutationally invariant norms on $\mathbb{R}^n$ and USI norms on $H_n$, namely, each USI norm $\| \cdot \|$ corresponds to a permutationally invariant norm $\Phi$ such that $\|A\| = \Phi(\lambda(A))$.

(a) For every USI norm on $H_n$, there is a compact set $E$ of real vectors $c = (c_1, \ldots, c_n)$ with entries arranged in descending order such that

$$\|A\| = \max \{ r_c(A) : c \in E \},$$

where

$$r_c(A) = \max \left\{ \sum_{j=1}^{n} c_j \lambda_j(A), -\sum_{j=1}^{n} c_{n-j+1} \lambda_j(A) \right\}$$

is the $c$-numerical radius of $A$. 

\[ \boxed{ \text{2} } \]
Let $A, B \in H_n$. Then $\|A\| \leq \|B\|$ for all USI norms $\| \cdot \|$ if and only if $r_c(A) \leq r_c(B)$ for a dense subset of

$$\left\{ (c_1, \ldots, c_n) : c_1 \geq \cdots \geq c_n, \sum_{j=1}^n c_j^2 = 1 \right\}.$$

**Definition 2.4** Let $C \in M_n$. Define the $C$-numerical radius of $A$ by

$$r_C(A) = \max \{|\text{tr} (CU^*AU)| : U \text{ is unitary}\},$$

and define the $C$-congruence numerical radius of $A$ by

$$\tilde{r}_C(A) = \max \{|\text{tr} (CU^tAU)| : U \text{ is unitary}\}.$$

**Theorem 2.5** (a) For every USI norm on $M_n$, there is a compact set $E$ of matrices

$$\|A\| = \max \{r_C(A) : C \in E\}.$$

(b) For every UCI norm on $M_n$, there is a compact set $E$ of matrices

$$\|A\| = \max \{\tilde{r}_C(A) : C \in E\}.$$

**Theorem 2.6** A norm $\| \cdot \|$ on $M_n$ is UI if and only if it is USI and UCI.

### 3 Best approximations

**Theorem 3.1** Let $\| \cdot \|$ be a UI norm on $M_n$.

(a) $\|A - (A + A^*)/2\| \leq \|A - H\|$ for any $H \in H_n$.

(b) $\|A - (A - A^*)/2\| \leq \|A - G\|$ for any $G \in iH_n$.

(c) If $A = UP$ for a unitary $U$ and a positive semi-definite $P$, then $\|A - U\| \leq \|A - V\|$ for any unitary $V \in M_n$.

**Theorem 3.2** Let $\| \cdot \|$ be a UI norm on $m \times n$ matrices.

(a) If $A = UDV$, where $U, V \in M_n$ are unitary and $D = \text{diag} (s_1, \ldots, s_n)$ such that $s_1 \geq \cdots \geq s_n \geq 0$, then $A_k = U\text{diag} (s_1, \ldots, s_k, 0, \ldots, 0)V$ satisfies $\|A - A_k\| \leq \|A - X\|$ for all $X$ with rank at most $k$.

(b) If $A = XDY^*$ so that $X$ is $m \times k$ with $X^*X = I_k$, $Y$ is $n \times k$ with $Y^*Y = I_k$, and $D = \text{diag} (s_1, \ldots, s_k)$ with $s_1 \geq \cdots \geq s_k > 0$, then for any given $m \times n$ matrix $B$ $\|A - UD^{-1}VB\| \leq \|A - X\|$ for all $m \times n$ matrix $X$. 

3
4 Norm bounds for sum and difference of matrices

Theorem 4.1 Let $\| \cdot \|$ be a UI norm on $M_n$. Suppose $A, B \in H_n$ have singular values $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$. Then

$$\frac{1}{\sqrt{2}} \| A + iB \| \leq \| \text{diag} (a_1 + ib_1, \ldots, a_n + ib_n) \| \leq \sqrt{2} \| A + iB \|.$$  

Theorem 4.2 Suppose $\| \cdot \|$ is a USI norm on $H_n$. If $A, B \in H_n$ have eigenvalues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, then

$$\| \text{diag} (a_1 - b_1, \ldots, a_n - b_n) \| \leq \| A - B \| \leq \| \text{diag} (a_1 - b_n, \ldots, a_n - b_1) \|.$$  

Theorem 4.3 Suppose $\| \cdot \|$ is a UI norm on $M_n$. If $A, B \in H_n$ have singular values $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, then

$$\| \text{diag} (a_1 - b_1, \ldots, a_n - b_n) \| \leq \| A - B \|.$$  

Theorem 4.4 Let $A, B \in M_n$, and let $\| \cdot \|$ be the operator norm. Then

$$\max\{\| U^*AU - V^*BV \| : U, V \text{ unitary} \} = \min\{\| A + \mu I \| + \| B + \mu I \| : \mu \in \mathbb{C} \}.$$  

The minimum on the right hand side will attain at a certain $\mu_0$ with $|\mu_0| \leq \max\{\| A \|, \| B \| \}$.  

5 Norm bounds involving product of matrices

Proposition 5.1 Suppose $A, B \in M_n$ are such that $AB \in H_n$.

(a) For any USI norm on $H_n$, $\| AB \| \leq \| \text{Re}(BA) \|$.  

(b) For any UI norm on $M_n$, $\| AB \| \leq \| \text{Re}(BA) \|$.  

Theorem 5.2 Let $\| \cdot \|$ be a UI norm on $M_n$. For any matrices $A, B, X$, we have

$$\| AXB^* \| \leq \frac{1}{2} \| A^*AX + XB^*B \|.$$  

Theorem 5.3 Let $\| \cdot \|$ be a UI norm. Suppose $A, B \in H_n$ are positive semi-definite. Then

$$4 \| AB \| \leq \| (A + B)^2 \|.$$  

Actually, we can prove

$$2s_j(A^{3/2}B^{1/2} + A^{1/2}B^{3/2}) \leq s_j(A + B)^2, \quad j = 1, \ldots, n.$$  

Consequently,

$$4 \| AB \| = 4\| A^{1/2}(A^{1/2}B^{1/2})B^{1/2} \| \leq 2\| A^{3/2}B^{1/2} + A^{1/2}B^{3/2} \| \leq \| (A + B) \|^2.$$  

4
6 Additional results and problems

Definition 6.1 For $p \geq 1$, define the Schatten $p$-norm on $M_n$ by

$$\|A\|_p = \left(\sum_{j=1}^{n} s_j(A)^p\right)^{1/p}.$$ 

Theorem 6.2 Let $A, B \in H_n$.
(a) If $1 \leq p \leq 2$, then

$$2^{2/p-1}\|A + iB\|_p^2 \geq \|A\|_p^2 + \|B\|_p^2 \geq 2^{1-2/p}\|A + iB\|_p^2.$$ 

(b) If $2 \leq p \leq \infty$, then

$$2^{2/p-1}\|A + iB\|_p^2 \leq \|A\|_p^2 + \|B\|_p^2 \leq 2^{1-2/p}\|A + iB\|_p^2.$$ 

Theorem 6.3 Let $A, B \in H_n$ be such that $A$ is positive semidefinite.
(a) If $1 \leq p \leq 2$, then

$$\|A + iB\|_p^2 \geq \|A\|_p^2 + 2^{1-2/p}\|B\|_p^2.$$ 

(b) If $2 \leq p \leq \infty$, then

$$\|A + iB\|_p^2 \leq \|A\|_p^2 + 2^{1-2/p}\|B\|_p^2.$$ 

(c) $\|A\|_2^2 + \|B\|_2^2 \leq \|A + iB\|_2^2$.

In addition, suppose $B$ is also positive semidefinite.

(d) If $1 \leq p \leq 2$, then

$$\|A + iB\|_p^2 \geq \|A\|_p^2 + \|B\|_p^2.$$ 

(e) If $2 \leq p \leq \infty$, then

$$\|A + iB\|_p^2 \leq \|A\|_p^2 + \|B\|_p^2.$$ 

Problem 6.4 Can one prove (d) and (e) without the assumption that $B$ is also positive semidefinite?

Theorem 6.5 (Böttcher and Wenzel) Let $A, B \in M_n$. Then

$$\|AB - BA\|_2 \leq \sqrt{2}\|A\|_2\|B\|_2.$$ 

Problem 6.6 Characterize $A$ and $B$ so that the equality holds.

It is known that if the equality holds, then $A$ and $B$ have rank at most 2, $A \in \{X \in M_n : BX = XB\}^\perp$ and $B \in \{X \in M_n : AX =XA\}^\perp$.

Problem 6.7 Prove or disprove that

$$\|AB - BA\|_p \leq 2^{1/\min(p,q)}\|A\|_p\|B\|_p.$$
7 Exercises

1. Show that every symmetric gauge function is Schur convex monotone.

2. Show that there exist a UI norm and $A, B \in M_n$ with singular values $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$ such that
   \[ \| A - B \| > \| \text{diag} (a_1 - b_n, \ldots, a_n - b_1) \|. \]

3. Suppose $\| \cdot \|$ is a UI norm on $M_n$. Let $A, B \in M_n$ have singular values $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, and $C = AB$ have singular values $c_1 \geq \cdots \geq c_n$. Prove the following.
   (a) For any $r > 0$,
   \[ (c^r_1, \ldots, c^r_n) \preceq_{\log} (a^r_1 b^r_1, \ldots, a^r_n b^r_n). \]
   (b) For any $p, q \geq 1$ such that $1/p + 1/q = 1$,
   \[ \| AB \| \leq \| A^p \|^{1/p} \| B^q \|^{1/q}. \]
   (c) For any $p, q > 0$ and $1/p + 1/q = 1/r$,
   \[ \| |AB|^r \|^{1/r} \leq \| |A|^p \|^{1/p} \| |B|^q \|^{1/q}. \]

   For $p = q = 2$, we have the Cauchy-Schwarz inequality for UI norms.

4. For any USI norm $\| \cdot \|$ on $M_n$, and $A = (A_{ij})_{1 \leq i, j \leq 2} \in M_n$ with $A_{11} \in M_m$ and $A_{22} \in M_{n-m}$, show that
   \[ \| A_{11} \oplus A_{22} \| \leq \| A \|. \]
   If $\| \cdot \|$ is UI, then $\| A_{11} \oplus 0_{n-m} \| \leq \| A \|$. What about USI norms?