**Math 150, Fall 2005** Jinchuan Hou and Chi-Kwong Li

**Problem** Consider an ice cream cone with height $2h$ and circular top with radius $r$. If one uses the *same material* (i.e., the same area of lateral surface) to make an ice cream cone with height $h$ and radius $R$, which cone has larger volume?

**Answer** It depends on $h$ and $r$.

Since the surface areas of the cones are the same,

$$2\pi r\sqrt{(2h)^2+r^2} = 2\pi R\sqrt{h^2+R^2}.$$

Thus,

$$r^2(4h^2+r^2) = R^2(h^2+R^2).$$

Rewrite the equation as:

$$R^4 + h^2R^2 - r^2(4h^2+r^2) = 0.$$

Solving for $R^2$, we use the quadratic formula and get

$$R^2 = \left[-h^2 + \sqrt{h^4+16r^2h^2+4r^4}\right]/2.$$

Hence, the volume of the two cones are:

$$\pi r^2(2h)/3 \quad \textit{and} \quad \pi R^2 h/3,$$

which has a ratio of

$$2r^2/R^2 = 4r^2/[\sqrt{h^4+16r^2h^2+4r^4} - h^2] = 4/[\sqrt{x^4+16x^2+4} - x^2],$$

where $x = h/r$. Hence, the ratio is less than 1 if and only if $x^4+16x^2+4 > (x^2+4)^2$, i.e., $x^2 > 3/2$.

**Example** If $r = h = 1$ unit, then the ratio is $4/(\sqrt{21}-1) > 1$, i.e., the first cone is larger. If $h = 2r = 2$ units, then the ratio is $4/(\sqrt{84}-4) < 1$, i.e., the second cone is larger.

**Remark** One has to be careful in extending the result on cylinders with similar bases.