§8 Groups of permutations (bijections)

Basic notation and ideas

We study the most “general type” of groups - groups of permutations (bijections).

Theorem (8.5) Let $A$ be a set, and let $S_A$ be the set of permutations on $A$, i.e., bijections from $A$ to $A$. Then $S_A$ is a group under function composition.

Definition (8.6) If $A$ has $n$ elements, we may assume $A = \{1, \ldots, n\}$, and write $S_A$ as $S_n$, the symmetric group of degree $n$, which has $n!$ elements.

Examples (Important) $S_1, S_2, S_3, S_4$.

Theorem (8.16) [Cayley] Every group $G$ is isomorphic to a subgroup of $S_G$.

Left and right regular representations of $G$ (2.1.17):

Let $G$ be a group. For every $g \in G$, define $\lambda_g : G \to G$ by $\lambda_g(x) = gx$. Then $\lambda_g \in S_G$.

Then the mapping $\phi : G \to S_G$ defined by $\phi(g) = \lambda_g$ define a isomorphism from $G$ to the subgroup $\phi[G]$ in $S_G$.

Here $\phi[G]$ is the left regular representation of $G$ in $S_G$. Similarly, the function $\psi : G \to S_G$ defined by $\psi(g) = \mu_g \in S_G$ such that $\mu_g(x) = xg^{-1}$ gives rise to the right regular representation $\psi[G]$ of $G$ in $S_G$.

§9 Orbits, Cycles, and the Alternating Groups

Definition (9.1) Let $\sigma \in S_A$. The orbit of $a \in A$ under $\sigma \in S_A$ is the set

$$O_{\sigma}(a) = \{\sigma^n(a) : n \in \mathbb{Z}\}.$$

The orbits of $\sigma \in S_A$ form a partition of $A$ because the following is an equivalence relation.

$$a \sim b \text{ in } A \text{ if } \sigma^n(a) = b \text{ for some integer } n.$$

Definition (9.6/9.11) A permutation $\sigma \in S_n$ is a cycle if it has at most one orbit containing more than one element. The length of the cycle is the number of elements in the largest orbit. Cycles of length two are transpositions.

Theorem (9.8/9.12/9.18) Consider the group $S_n$.

(a) Every $\sigma \in S_n$ is a product of disjoint cycles.

(b) Every length $k$ cycle, where $k \geq 2$, is a product of $k - 1$ transpositions.

(c) Every permutation is a product of transpositions (may not be disjoint).

(d) No permutation can be expressed as a product of even number of transpositions and as a product of a product of odd number of transpositions.

Definition (9.18) A permutation is even (odd) if it can be written as the product of an even (odd) number of transpositions.

Theorem (9.20) The set of even permutations in $S_n$ form a group $A_n$ with $n!/2$ so many elements. It is called the alternating group on $n$ letters (Definition 9.21).