# The Ninth Workshop on Numerical Ranges and Numerical Radii

Department of Mathematics, College of William and Mary,
July 19 (Saturday) - July 21 (Monday), 2008.

Endorsed by The International Linear Algebra Society, and

Sponsored by The College of William and Mary.

Coffee breaks will be at the first floor of Jones Hall.
Workshop dinner on July 20 will be Chinese buffet.
Workshop photo on July 21 will be taken outside Jones Hall.
IWOTA reception on July 21 will be at the University Center, Lodge 1.

## Schedule

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**July 19 (Saturday)**

1:30 - 2:00 Man-Duen Choi, University of Toronto.
The numerical ranges of powers of operators

2:00 - 2:30 Ilya Spitkovsky, The College of William & Mary.
Numerical ranges of certain quadratic operators

2:30 - 3:00 Boris Mirman, Suffolk University.
Matrices that generate separated Poncelet's curves

3:00 - 3:30 **Coffee break**

3:30 - 4:00 Leiba Rodman, The College of William & Mary.
Preservers of spectral radius, numerical radius, or spectral norm of the sum on nonnegative matrices

4:00 - 4:30 Tin-Yau Tam, Auburn University.
Generalized numerical ranges and complex semisimple Lie algebras

4:30 - 5:00 Ricardo E. Teixeira, CMUC/University of Azores.
Flat portions on the boundary of the indefinite numerical range of $3 \times 3$ matrices

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**July 20 (Sunday)**

1:30 - 2:00 David Kribs, Department of Mathematics & Statistics, University of Guelph and Institute for Quantum Computing, University of Waterloo
On numerical range techniques in quantum computing

2:00 - 2:30 Hwa-Long Gau, National Central University, Taiwan.
Higher-Rank Numerical Ranges and Unitary Equivalence

2:30 - 3:00 Yiu-Tung Poon, Iowa State University
Geometrical properties of joint higher rank numerical range

3:00 - 3:30 **Coffee break**

3:3- - 4:00 Nung-Sing Sze, University of Connecticut.
Higher rank numerical ranges and low rank perturbations of quantum channels

4:00 - 4:30 Sean Clark, The College of William & Mary.
Linear preservers of the higher rank numerical range and radius

4:30 - 5:00 Jennifer Mahle, The College of William & Mary.
Preservers of the joint higher rank numerical range

6:00 **Workshop Dinner**

**July 21 (Monday)**

9:30-10:00 Thomas Schulte-Herbrüggen, Technical University Munich, Garching, Germany.
Ranges in Quantum Dynamics: Foundations and
Applications of New Methods for Tensor SVD

10:00 - 10:30 Mao-Ting Chien,Soochow University, Taiwan.
Commutativity of $C$-numerical range

10:30 - 11:00 **Coffee break**

11:00 - 11:30 Panayiotis Psarrakos, National Technical Univ. of Athens.
A definition of numerical range based on the Birkhoff-James orthogonality

11:30 - noon Christiane Tretter, University of Bern.
Quadratic numerical range (QNR) of analytic block operator matrix functions

noon - 12:30 Jinchuan Hou, University of Technology, P. R. China,
Maps preserving numerical radius of operator products on $\mathcal{S}(H)$

12:30 - 1:30 **Pizza lunch and Workshop Photo**

1:30 - 2:00 Pizza lunch continued.

2:00 - 2:30 Ana Nata, The Polytechnic Institute of Tomar, Portugal.
Krein Spaces Numerical Ranges and their computer generation

2:30 - 3:00 Paul Zachlin, Lakeland Community College.
Eigenvalue inclusion regions from inverses of shifted matrices

3:00 - 3:30 **Coffee break**

3:30 - 4:00 F.H. Szafraniec, Uniwersytet Jagielloński, Kraków, Poland.
Extending the class of $\mathcal{C}_\rho$ operators

4:00 - 4:30 Hocine Guediri, King Saud University, Saudi Arabia.
The Numerical Range of a Dual Toeplitz Operator

4:30 - 5:00 Chi-Kwong Li, The College of William and Mary.
Elliptical range theorems for generalized numerical ranges of quadratic operators

7:00 - 10:00 **IWOTA Reception**

## Titles and Abstracts

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### On Furata's inequality of indefinite type

**Speaker** Natália Bebiano, University of Coimbra, bebiano@mat.uc.pt

**Abstract** A selfadjoint involutive matrix $J$ endows $\mathbf{C}^n$ with an indefinite inner product $[\cdot,\cdot]$ given by $[x,y] := \langle Jx, y\rangle$, $x, y \in \mathbf{C}^n$. Some exponential operator inequalities for $J-$selfadjoint matrices are presented. The $J-$chaotic order is characterized in terms of operator functions involving the $\alpha-$power mean. Our main result is an indefinite complete form of Furuta inequality for $J-$contractions (or $J-$expansions). The parallelism between the definite versions of the inequalities in Hilbert spaces and the corresponding indefinite versions in Krein spaces is pointed out.

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### Commutativity of $C$-numerical range

**Speaker** Mao-Ting Chien, Soochow University, Taiwan. mtchien@scu.edu.tw.

**Co-author** Hiroshi Nakazato, Hirosaki University, Japan

**Abstract** Let $A$ and $C$ be $n$-by-$n$ complex matrices. The $C$-numerical range of $A$ is defined to be the set

$$W_C(A) = \{\mathrm{tr}(CU^*AU) : U \in M_n, U^*U = I_n\}.$$

We study classes of matrices that two matrices $A$, $B$ in the respective class satisfy $W_C(AB) = W_C(BA)$ for certain complex matrix $C$. The classes include some 2-by-2 and 3-by-3 matrices, $n$-by-$n$ symmetric matrices, Toeplitz matrices and continuant matrices.

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### The numerical ranges of powers of operators

**Speaker** Man-Duen Choi, University of Toronto, choi@math.wm.edu

**Abstract** The location of the numerical ranges of integer powers of an operator will be discussed.

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### Linear Preservers of the higher rank numerical range and radius

**Speaker** Sean Clark, College of William and Mary, siclar@wm.edu

**Co-authors** Chi-Kwong Li, Jennifer Mahle and Leiba Rodman, College of William and Mary.

**Abstract** It is shown that linear maps preserving the higher rank numerical range on $n \times n$ matrices have the form

$$A \mapsto U^*AU \qquad \text{or} \qquad A \mapsto U^*A^tU$$

for some unitary matrix $U$. Moreover, it is shown that a linear map preserving the higher rank numerical radius must be a unit multiple a preserver of the higher rank numerical range.

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## Higher-Rank Numerical Ranges and Unitary Equivalence

**Speaker** Hwa-Long Gau, National Central University, Taiwan, hlgau@math.ncu.edu.tw.

**Co-authors** Pei Yuan Wu, National Chiao Tung University.

**Abstract** Denote by $\Lambda_k(A) = \{\lambda \in \mathbf{C} : X^*AX = \lambda I_k$ for some $n$-by-$k$ $X$ satisfies $X^*X = I_k\}$ the rank-$k$ numerical range of an $n$-by-$n$ complex matrix $A$. We show that if $A$ and $B$ are two normal (or companion) matrices, then $A$ and $B$ are unitarily equivalent if and only if $\Lambda_k(A) = \Lambda_k(B)$ for all $k = 1, \ldots, n$. However, the more general assertion that an $n$-by-$n$ matrix is determined by its higher-rank numerical range turns out to be false. We also show, for $n$-by-$n$ matrices $A$ and $B$, $\Lambda_k(A) = \Lambda_k(B)$ for all $k = 1, \ldots, n$ if and only if $p_A(x, y, z) = p_B(x, y, z)$, where $p_A$ is the degree-$n$ homogeneous polynomial in $x, y$ and $z$ given by $p_A = \det(x(A + A^*)/2 + y(A - A^*)/(2i) + zI_n)$.

## The Numerical Range of a Dual Toeplitz Operator

**Speaker** Hocine Guediri, King Saud University, Saudi Arabia, hguediri@ksu.edu.sa

**Abstract** Let $D$ be the unit disk in the complex plane and let $dA(z)$ be the Lebesgue area measure on $D$. The Bergman space $L^2_a$ is the Hilbert subspace of $L^2(D, dA)$ consisting of analytic functions. The orthogonal complement of $L^2_a$ in $L^2(D, dA)$ is denoted by $(L^2_a)^\perp$. A dual Toeplitz operator is defined on $(L^2_a)^\perp$ to be a multiplication followed by a projection onto $(L^2_a)^\perp$.

We are concerned with qualitative properties of the numerical range of a dual Toeplitz operator. We consider various classes of such operators, such as normal and quasinormal, as well as more general ones. We completely characterize the numerical range of some of them and establish main qualitative properties of others. As a byproduct, we establish more corresponding spectral properties; further we shed some light on the analog of Halmos' classification problem of subnormal Toeplitz operators.

## Maps preserving numerical radius of operator products on $\mathcal{S}(H)$

**Spearker** Jinchuan Hou, University of Technology, P. R. China, jinchuanhou@yahoo.com.cn

**Co-author** Kan He, Shanxi Normal University, Linfen, Shanxi 041004, P. R. China.

**Abstract** Let $H$ be a complex Hilbert space with $\dim H \geq 3$, $\mathcal{S}(H)$ the (real) Jordan algebra of all self adjoint operators on $H$. Every surjective map $\Phi : \mathcal{S}(H) \to \mathcal{S}(H)$ preserving numerical radius of operator products (respectively, Jordan triple products) is characterized. It is shown that $w(\Phi(A)\Phi(B)) = w(AB)$ (respectively, $w(\Phi(B)\Phi(A)\Phi(B)) = w(BAB)$) for all $A, B \in \mathcal{S}(H)$ if and only if there exist a unitary or a conjugate unitary operator $U$ and a functional $h : \mathcal{S}(H) \to \{-1, 1\}$ such that $\Phi(A) = h(A)UAU^*$ for all $A \in \mathcal{S}(H)$.

**On numerical range techniques in quantum computing**

**Speaker** David Kribs, Department of Mathematics & Statistics, University of Guelph and Institute for Quantum Computing, University of Waterloo, dkribs@uoguelph.ca

**Abstract** In this talk I will discuss some instances in quantum computing where numerical range techniques arise. I will also try to formulate some open problems.

**Elliptical range theorems for generalized numerical ranges of quadratic operators**

**Speaker** Chi-Kwong Li, William and Mary, ckli@math.wm.edu

**Co-authors** Yiu-Tung Poon, Iowa State University, ytpoon@iastate.edu; Nung-Sing Sze, University of Connecticut, sze@math.uconn.edu

**Abstract** The classical numerical range of a quadratic operator is an elliptical disk. This result is extended to different kinds of generalized numerical ranges. In particular, it is shown that for a given quadratic operator, the rank-$k$ numerical range, the essential numerical range, and the $q$-numerical range are elliptical disks; the $c$-numerical range is a sum of elliptical disks, and the Davis-Wielandt shell is an ellipsoid with or without interior.

**Preservers of the joint higher rank numerical range**

**Speaker** Jennifer Mahle, College of William and Mary, jrmahle@wm.edu

**Co-authors** Sean Clark and Chi-Kwong Li, College of William and Mary.

**Abstract** It is shown that linear preservers of the joint higher rank numerical range of $m$-tuples of square matrices have the form

$$(A_1, \ldots, A_m) \mapsto (U^* A_1 U, \ldots, U^* A_m U) \quad \text{or} \quad (A_1, \ldots, A_m) \mapsto (U^* A_1^t U, \ldots, U^* A_m^t U)$$

for some unitary matrix $U$. Moreover, it is shown that that the linearity assumption can be replaced by additivity and surjectivity. To achieve this, we show that additive maps mapping the cone of positive semi-definite matrices onto itself must be linear.

**Matrices that generate separated Poncelet's curves**

**Speaker** Boris Mirman, Suffolk University, bmirman@rcn.com

**Abstract** The author continues exploring Poncelet curves, leveraging their connection to the numerical ranges of special matrices. This connection was recently found by Gau and Wu (1998) and Mirman (1998). Here, Poncelets porism is considered in the real plane. Besides the nested and intersected Poncelets curves, there may be separated curves. Namely: a polygon inscribed in a conic may be such that the continuations of all its sides are tangent to a curve which is separated from the conic. For this case, an alternating in sign function and the corresponding matrix that generate the Poncelets curves are presented. That is instead of a measure density function and the corresponding UB-matrix for the nested case. Examples demonstrate the possible shape and location of Poncelets curves and Poncelets polygons. The general equations are applied to the case of Poncelets conics. A recursive procedure for foci of a Poncelets package and the corresponding matrices of the focis numbers are presented. These integer- valued matrices are invariants: they depend only on the number of polygon sides. Even not knowing this number one can sometimes conclude whether a closed Poncelets polygon does not exist for a given pair of conics. Based on the properties of the integer-valued matrices, it is easier to get some formulas usually derived from elliptic addition rule. Considerations started by Steiner (1827), Richelot (1848), Cayley (1853-1861), Clifford (1867-1868), Wolstenholme (1876), Titchmarsh (1922), Chaundy (1923-1926) are continued and partially simplified.

**Krein Spaces Numerical Ranges and their computer generation**

**Speaker** Ana Nata, Department of Mathematics of the Polytechnic Institute of Tomar, Portugal, anata@ipt.pt

**Co-author 1** Natália Bebiano, University of Coimbra, bebiano@mat.uc.pt
**Co-author 2** João da Providência, University of Coimbra, providencia@teor.fis.uc.pt
**Co-author 3** Graça Soares Trás-os-Montes e Alto Douro, gsoares@utad.pt

**Abstract** Let $J$ be an involutive Hermitian matrix with signature $(t, n-t)$, $0 \leq t \leq n$, that is, with $t$ positive and $n-t$ negative eigenvalues. The Krein space numerical range of a complex matrix $A$ of size $n$ is denoted by $W_J(A)$ and is the collection of complex numbers of the form $\frac{x^*JAx}{x^*Jx}$, with $x \in \mathbf{C}^n$ and $x^*Jx \neq 0$.

Since $W_J(A)$ is, in general, neither bounded nor closed, the description of this set may be complicated and so it is of interest to have a code to produce its graphical representation. An algorithm and respective *Matlab* program to generate Krein spaces numerical ranges of arbitrary complex matrices that treats the degenerate cases and represents the boundary generating curves is given. We emphasize that it also works for Hilbert spaces numerical ranges. The routines of our program are available at: http://www.mat.uc.pt/∼bebiano

Moreover, a class of tridiagonal matrices, that is, matrices $A = (a_{ij})$ such that $a_{ij} = 0$ whenever $|i-j| > 1$, are considered. Interesting papers have been published on the classical numerical range of tridiagonal matrices. Likewise, there is interest in studding Krein space numerical ranges of this class of matrices. Namely, we characterize the $J-$numerical range of tridiagonal matrices $A = (a_{ij})$ with biperiodic main diagonal, that is, $a_{jj} = a_1$ if $j$ is odd and $a_{jj} = a_2$ if $j$ is even, and with $b'$s on the first superdiagonal and $c'$s on the first subdiagonal such that either $c_j = k\bar{b}_j$ or $b_j = k\bar{c}_j$ for some $k \in \mathbf{C}$ and $j = 1, \ldots, n-1$.

**Geometrical properties of joint higher rank numerical range**

**Speaker** Yiu-Tung Poon, Iowa State University, ytpoon@iastate.edu

**Co-authors** Chi-Kwong Li, William and Mary, ckli@math.wm.edu; Nung-Sing Sze, University of Connecticut, sze@math.uconn.edu

**Abstract** Geometrical properties of the joint higher rank numerical ranges of Hermitian operators are studied. It is shown that if the dimension of the operators is large enough, their joint rank k-numerical range is always star-shaped. For infinite dimensional operators, the connection between the joint higher rank numerical ranges and the essential joint numerical range of the operators is also discussed.

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**A definition of numerical range based on the Birkhoff-James orthogonality**

**Speaker** Panayiotis Psarrakos, National Technical Univ. of Athens, (ppsarr@math.ntua.gr)

**Co-authors** Ch. Chorianopoulos and S. Karanasios

**Abstract** The numerical range of an operator can be written as an (infinite) intersection of closed circular discs. This interesting property was observed by Bonsall and Duncan (1973), but it does not seem to be very popular to people working on numerical ranges. In this paper, we propose a new simple proof that is based on the properties of norms and the Birkhoff-James orthogonality. Furthermore, our approach leads to the introduction of a definition of numerical range of rectangular complex matrices. This new range is always compact and convex, and satisfies basic properties of the standard numerical range.

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**Preservers of spectral radius, numerical radius, or spectral norm of the sum on nonnegative matrices**

**Speaker** Leiba Rodman, The College of William and Mary, lxrodm@math.wm.edu.

**Co-author** Chi-Kwong Li, The College of William and Mary, ckli@math.wm.edu

**Abstract** We characterize maps $f$ on the set of real square size matrices with nonnegative entries such that $r((f(A) + f(B)) = r(A + B)$ for all such matrices $A$ and $B$. Here $r(X)$ is the spectral radius of $X$. No a priori hypotheses, such as linearity of the map etc., are assumed. It turns out that all such maps are given by either similarity or similarity followed by transposition, where the similarity matrix and its inverse have nonnegative entries. Moreover, the same conclusion holds if the spectral radius is replaced by the spectrum or the peripheral spectrum. Similar results are obtained for maps on the set of nonnegative symmetric matrices. Furthermore, we obtain descriptions of all maps $f$ with the property that $w((f(A)+f(B)) = w(A+B)$ for all nonnegative matrices $A$ and $B$, or with the property that $\|f(A)+f(B)\| = \|A+B\|$ for all such matrices $A$ and $B$. Here $w(X)$ is the numerical radius and $\|X\|$ is the spectral norm of $X$, respectively. In the case of the numerical radius, it turns out that the standard expected forms do not describe all such maps $f$.

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**Numerical ranges of certain quadratic operators**

**Speaker** Ilya Spitkovsky, The College of William & Mary, ilya@math.wm.edu

**Coauthor** Leiba Rodman, The College of William & Mary, lxrodm@math.wm.edu

**Abstract** According to Tso-Wu 1999 result, the numerical range of a quadratic operator is an ellipse. We show that this is also true for the essential numerical range, and find sufficient conditions when the $c$-numerical range is also an ellipse. These abstract results are illustrated by several examples of singular integral operators and composition operators on Lebesgue spaces (weighted or not) and Dirichlet spaces. For these examples, explicit formulas for the parameters of the ellipses in question are given.

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**$C$-Numerical Ranges in Quantum Dynamics: Foundations and Applications of New Methods for Tensor SVD**

**Speaker** Thomas Schulte-Herbrüggen, Technical University Munich (TUM), Garching, Germany, tosh@ch.tum.de

**Co-author** Gunther Dirr & Uwe Helmke, University of Würzburg, Germany

**Abstract** In the 2006 WONRA we introduced the *relative $C$-numerical range* as the new entity $W_{\mathbf{K}}(C,A) := \{\mathrm{tr}(C^\dagger KAK^\dagger) | K \in \mathbf{K}\}$, where $\mathbf{K}$ is a compact connected subgroup to the unitary group [LAMA 56, 3 and 27 (2008)].— We now focus on new applications.

Least-squares approximations $\min_K ||KAK^\dagger - C||_2^2$ are determined by the maximum real part obtainable within $W_{\mathbf{K}}(C,A)$. In view of applications in quantum dynamics, here we show new gradient flows on the unitary orbit $\mathcal{O}_{\mathbf{U}}(A) := \{UAU^\dagger | U \in \mathbf{U}\}$ as well as the relative unitary orbit $\mathcal{O}_{\mathbf{K}}(A) := \{KAK^\dagger | K \in \mathbf{K} \subset \mathbf{U}\}$ that can be taken to approximate a given operator $C$ by points on the (relative) unitary orbit of $A$, where $A$ takes the form of a rank-$p$ projector. Even the simplest case of $A, C$ both being rank" 1 projectors is non-trivial as soon as the relative orbit is restricted to a proper subgroup of the full unitary group. For illustration, we focus on the case $\mathbf{K} = SU(2)^{\otimes n} \subset SU(2^n),\ n \geq 2$. For rank-1 approximations, the new gradient flows on the relative unitary orbit provide powerful alternatives with higher speed and reliability compared to established algorithms of finding the tensor SVDs via higher-order power methods (HOPM) or higher-order iteration (HOOI).

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**Extending the class of $\mathcal{C}_\rho$ operators**

**Speaker** F.H. Szafraniec, Uniwersytet Jagielloński, Kraków, Poland; umszafra@cyf-kr.edu.pl

**Abstract** The operators having unitary $\rho$ dilation are known to be well situated in the numerical range environment. Taking this as an excuse I intend to breathe new life into the substance of my old paper

F.H. Szafraniec, Orthogonal decompositions of non-contractive operator valued representations of Banach algebras, *Bull. Acad. Polon. Sci., Sér. sci. math. astr. et phys.*, **19**(1971), 937-940.

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**Higher rank numerical ranges and low rank perturbations of quantum channels**

**Speaker** Nung-Sing Sze, University of Connecticut, sze@math.uconn.edu

**Co-authors** Chi-Kwong Li, William and Mary, ckli@math.wm.edu; Yiu-Tung Poon, Iowa State University, ytpoon@iastate.edu

**Abstract** For a positive integer $k$, the rank-$k$ numerical range $\Lambda_k(A)$ of an operator $A$ acting on a Hilbert space $\mathcal{H}$ of dimension at least $k$ is the set of scalars $\lambda$ such that $PAP = \lambda P$ for some rank $k$ orthogonal projection $P$.

In this talk, the connection between $\Lambda_k(A)$ and $\Lambda_{k-r}(A+F)$, the rank-$(k-r)$ numerical range of $A$ with a perturbation of a rank $r$ operator $F$, will be discussed. In particular, it can be shown that if $A$ is normal or if the dimension of $A$ is finite, then $\Lambda_k(A)$ can be obtained as the intersection of $\Lambda_{k-r}(A+F)$ for a collection of rank $r$ operators $F$.

Furthermore, results for the rank-$\infty$ numerical range $\Lambda_\infty(A)$ will also be studied, where $\Lambda_\infty(A)$ is defined as the set of scalars $\lambda$ such that $PAP = \lambda P$ for an infinite rank orthogonal projection $P$.

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**Generalized numerical ranges and complex semisimple Lie algebras**

**Speaker** Tin-Yau Tam, Auburn University timtiny@auburn.edu.

**Abstract** We discuss how the complex semisimple Lie algebras/groups come into the scene of the study of generalized numerical ranges. Convexity and star-shapedness results will be discussed.

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### Flat portions on the boundary of the indefinite numerical range of $3 \times 3$ matrices

**Author** Ricardo E. Teixeira, CMUC/University of Azores, Portugal, rteixeira@uac.pt

**Co-authors** N. Bebiano, CMUC/University of Coimbra, Portugal, bebiano@mat.uc.pt and J. da Providência, Physics, University of Coimbra, Portugal, providencia@teor.fis.uc.pt

**Abstract** For $J = I_r \oplus -I_{n-r}$ $(0 < r < n)$, where $I_m$ denotes the identity matrix of order $m$, consider $\mathbf{C}^n$ endowed with the Krein structure defined by the indefinite inner product $\langle \xi_1, \xi_2 \rangle_J = \xi_2^* J \xi_1,\ \xi_1, \xi_2 \in \mathbf{C}^n$. Let $M_n$ be the algebra of $n \times n$ complex matrices. The *J-numerical range* of $A \in M_n$ is defined as

$$W_J(A) = \left\{ \frac{\xi^* J A \xi}{\xi^* J \xi} : \xi \in \mathbf{C}^n,\, \xi^* J \xi \neq 0 \right\}.$$

We derive canonical forms for $3 \times 3$ irreducible matrices with a flat portion on the boundary of the indefinite numerical range. We investigate $W_J(A)$ for upper triangular matrices $A \in M_3$. The particularly simple case of triangular matrices with one-point spectrum is discussed. The results here obtained are parallel to those of Keeler, Rodman and Spitkovsky for the classical numerical range.

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### Quadratic numerical range (QNR) of analytic block operator matrix functions

**Speaker** Christiane Tretter, University of Bern, tretter@math.unibe.ch

**Abstract** We extend the recently introduced concept of quadratic numerical range of block operator matrices to analytic block operator matrix functions. The main results include the spectral inclusion property and resolvent estimates.

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### Eigenvalue inclusion regions from inverses of shifted matrices

**Speaker** Paul Zachlin, Lakeland Community College, pzachlin@lakelandcc.edu

**Co-authors** Michiel Hochstenbach, TU Eindhoven, and David Singer, Case Western Reserve University

**Abstract** We consider eigenvalue inclusion regions based on the field of values of the inverse of a shifted matrix. A family of these inclusion regions is derived by varying the shift. We study several properties, one of which is that the intersection of a family is exactly the spectrum. We also obtain results by using eigenvalue inclusion regions other than the field of values, such as Gershgorin regions, Brauer regions, and pseudospectra.

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