Sums, series, sequences

(From Calculus II)
\[ \sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \]

Geometric series:
\[ a_n = a_1 \cdot r^{n-1}, \quad S_n = \sum_{i=1}^{n} a_i = \frac{a_1(1 - r^n)}{1 - r}, \quad S = \sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} S_n = \frac{a_1}{1 - r} \]

\[ p\text{-series: } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ convergent when } p > 1, \text{ divergent when } p \leq 1 \]

(Taylor series) \( f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \), (Maclaurin series) \( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n \)

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots, \quad \ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \]

\[ \ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \]

\[ \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]

\[ \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \]

Methods of finding sums: telescope series, differentiate and integrate the power series

Examples:

1. (1) Find a formula for \( \sum_{k=1}^{n} k^3 \); (2) Show that \( \sum_{n=1}^{99} \frac{1}{\sqrt{n} + \sqrt{n+1}} = 9. \)

2. Find the sum \( \cos(\theta) + \cos(2\theta) + \cos(3\theta) + \cdots + \cos(n\theta) \). (Hint: use Euler’s formula: \( e^{ix} = \cos(x) + i\sin(x) \))

3. (UIUC 2003) Evaluate \( \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots \)

4. (1) \( \sum_{n=0}^{\infty} \frac{(n+1)^2}{3^n} \); (2) \( \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} \); (3) Evaluate \( \sum_{r=1}^{\infty} \left( \sum_{s=1}^{r} s^2 \right)^{-1} \)
