Recurrent Sequences

(difference equations and differential equations)

Quick review of Math 302 (Differential Equations):

$$y'' = Ay' + By, y(0) = a_0, y'(0) = a_1, \tag{1}$$

for which the solution is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \tag{2}$$

where λ_1 and λ_2 are the roots of quadratic characteristic equation $\lambda^2 = A\lambda + B$. If $\lambda_1 = \lambda_2$, then the solution is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}. \tag{3}$$

Recurrence relations

A recurrent sequence $\{x_n\}$ is defined as

$$x_{n+k} = f(x_n, x_{n+1}, x_{n+2}, \cdots, x_{n+k-1}),\ x_0 = a_0,\ x_1 = a_1, \cdots, x_{k-1} = a_{k-1}. \tag{4}$$

(4) is also called a *difference equation*. The number k is the order of the equation (or relation). A k-th order linear recurrent sequence is generated by a linear equation:

$$\begin{aligned} &x_{n+k} = b_n x_n + b_{n+1} x_{n+1} + b_{n+2} x_{n+2} + \cdots + b_{n+k-1} x_{n+k-1}, \\ &x_0 = a_0, x_1 = a_1, \cdots, x_{k-1} = a_{k-1}. \end{aligned} \tag{5}$$

The most common ones are first order recurrent sequence:

$$x_{n+1} = f(n, x_n), x_0 = a_0, \tag{6}$$

or second order recurrent sequence:

$$x_{n+2} = f(n, x_n, x_{n+1}), x_0 = a_0, x_1 = a_1. \tag{7}$$

An autonomous first order recurrent sequence:

$$x_{n+1} = f(x_n), x_0 = a_0, \tag{8}$$

is also often called a map.

Linear recurrence relations

The theory of linear recurrent sequence is very similar to that of linear ordinary differential equation. For example, a second order linear recurrent sequence:

$$x_{n+2} = Ax_n + Bx_{n+1}, x_0 = a_0, x_1 = a_1. \tag{9}$$

The solution is given by

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n, \tag{10}$$

where λ_1 and λ_2 are the roots of quadratic characteristic equation $\lambda^2 = A\lambda + B$, and c_1, c_2 are to be determined by the initial conditions. If $\lambda_1 = \lambda_2$, then the solution is

$$x_n = c_1\lambda_1^n + c_2 n\lambda_1^n. \tag{11}$$

For non-homogeneous linear recurrent sequence:

$$x_{n+2} = Ax_n + Bx_{n+1} + C, x_0 = a_0, x_1 = a_1, \tag{12}$$

Notice that it is possible to make a change of variable $y_n = x_n + k$, so that y_n satisfies (9), and k is determined by A, B and C.

Nonlinear recurrence relations

Since linear autonomous recurrent equations always have solution formulas, most problems in mathematics competitions are either non-autonomous or nonlinear. However the methods for linear equation are still very useful, and sometimes non-autonomous or nonlinear maybe reduced to linear autonomous equation via certain smart change of variables. In general there is no explicit solution formula for non-autonomous or nonlinear equation, even for a simple equation like $x_{n+1} = Ax_n(1 - x_n)$ (Logistic equation). Indeed the solutions of logistic equation are chaotic when the parameter A is large. (If you have the textbook of Math 302 (Blanchard-Devaney-Hall: Differential Equations), Chapter 8 of that book has a good introduction for logistic equation.)

For the 1st order nonlinear autonomous recurrent equation $x_{n+1} = f(x_n)$, a fixed point x is the one satisfying $x = f(x)$. Note that for such equation, $x_{n+1} = f^n(x_0)$, where $f^n(y) = f(f^{n-1}(y))$. So it is also often called iterated sequence. A fixed point x is attracting if $f^n(y) \to x$ for all y near x, and it is repelling if $f^n(y)$ goes away from x for all y near x. (There are also fixed point neither attracting nor repelling.) A fixed point x is attracting if $|f'(x)| < 1$, and it is repelling if $|f'(x)| > 1$. The iterated sequence can be drawn in $x - y$ coordinate system with so-called web diagram.

Finally, one can have a system of difference equations:

$$x_{n+1} = Ax_n + By_n, \;\; y_{n+1} = Cx_n + Dy_n, \tag{13}$$

and the solution is given in a form $(x_n, y_n) = (c_1, c_2)\lambda_1^n + (c_3, c_4)\lambda_2^n$, and λ_1, λ_2 are the eigenvalues of the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{14}$$

Some web-links on difference equations:

http://hypatia.math.uri.edu/~kulenm/diffeqaturi/dehomepage.html
http://www.math.duke.edu/education/ccp/materials/linalg/diffeqs/contents.html

Linear Fractional Transformations

Let $F(x) = \dfrac{ax+b}{cx+d}$, $G(x) = \dfrac{ex+f}{gx+h}$. Show that $F \circ G(x) = \dfrac{Ax+B}{Cx+D}$, where the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the multiplication of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$. (In advanced math language, it means the composition operation of the class of the functions $F(x) = (ax+b)/(cx+b)$ is equivalent to the matrix multiplication of 2×2 matrices, and the group of the class of the functions $F(x) = (ax+b)/(cx+b)$ is homeomorphic to the group of 2×2 matrices.) This fact is useful for recurrence relation:

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}. \tag{15}$$

See more from http://www.mathpages.com/home/kmath464/kmath464.htm

Problems on recurrent relations:

1. Find general solution of 1st order equation: $x_{n+1} = Ax_n$, $x_0 = a_0$.
2. Find general solution of 1st order equation: $x_{n+1} = Ax_n + C$, $x_0 = a_0$.
3. Find the solution of Fibonacci sequence: $x_{n+2} = x_{n+1} + x_n$, $x_0 = x_1 = 1$.
4. (UIUC 2004) Define a sequence $\{a_n\}$ by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + a_{n-2} - a_{n-3} + 1$ for $n \geq 3$. Find, with proof, a_{2004}.
5. (UIUC 1998) A sequence $a_0, a_1, a_2, \cdots$ of real numbers is defined recursively by $a_0 = 1$, $a_{n+1} = \dfrac{a_n}{1 + na_n}$, $n = 0, 1, 2, \cdots$. Find a general formula of a_n.
6. (UIUC 2003 Mock) Given $x_0 = 0$, define $x_{k+1} = \dfrac{x_k^2 - 2}{2x_k - 3}$. Determine if the sequence $\{x_n\}$ is convergent and if it is, find its limit.
7. (VT 1980) Given the linear fractional transformation $f_1(x) = (2x-1)/(x+1)$, define $f_{n+1}(x) = f_1(f_n(x))$ for $n = 1, 2, 3, \cdots$ It can be shown that $f_{35} = f_5$. Determine A, B, C, and D so that $f_{28}(x) = (Ax+B)/(Cx+D)$.
8. (VT 2001) Find a function $f(x) : \mathbf{R}^+ \to \mathbf{R}^+$ such that $f(f(x)) = \dfrac{3x+1}{x+3}$ for all positive real number x.