Problem Set 9

Discussion: Nov. 3, Nov. 8, Nov. 10 (on probability and binomial coefficients) The name after the problem is the designated writer of the solution of that problem. (No one is exempted these two weeks)

Discussion Problems

1. (Putnam 1989-B1) A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\dfrac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers. (`Beth`)

2. How many 4 digit numbers are there (consisting of the digits 0 through 9) with no digit appearing exactly two times. (`Derek`)

3. A one-foot stick is broken at random in two places. What's the average length of the smallest piece? Middle piece? Largest piece? (`Frank`)

4. Two points are picked at random on the unit circle $x^2+y^2 = 1$. What is the probability that the chord joining the two points has length at least 1? (`Erin`)

5. On a $m \times n$ checker board, choose two squares so that they are not in the same row or column. How many different choice do you have? (`Brett`)

6. (Putnam 1961) Let α and β be given positive real numbers with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α? (`David Edmonson`)

7. Pepys wrote Newton to ask which of three events is more likely: that a person get (a) at least 1 six when 6 dice are rolled, (b) at least 2 sixes when 12 dice are rolled, or (c) at least 3 sixes when 18 dice are rolled. What is the answer? (`Ben`)

8. Slips of paper with the numbers from 1 to 99 are placed in a hat. Five numbers are randomly drawn out of the hat one at a time (without replacement). What is the probability that the numbers are chosen in increasing order? (`Tina`)

9. In how many ways can n be written as a sum of k nonnegative integers, if the order is taken into account (so that, for example, $10 = 3 + 3 + 4$ and $10 = 3 + 4 + 3$ count as different representations)? (`Lei`)

10. (Putnam 1993-B3) Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers. (`Shelley`)

11. (Putnam 1992-B2) For nonnegative integers n and k, define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.) (`Nicholas`)

12. (Putnam 1958-B3) Real numbers are chosen at random from the interval $[0, 1]$. If after choosing the n-th number the sum of the numbers so chosen first exceeds 1, show that the expected or average value for n is e. (`Richard`)

13. (MIT training problem) Three closed boxes lie on a table. One box (you don't know which) contains a \$1000 bill. The others are empty. After paying an entry fee, you play the following game with the owner of the boxes: you point to a box but do not open it; the owner then opens one of the two remaining boxes and shows you that it is empty; you may now open either the box you first pointed to or else the other unopened box, but not both. If you find the \$1000, you get to keep it. Does it make any difference which box you choose? What is a fair entry fee for this game? (Frank has discussed this problem in his presentation, but here you also need to calculate the expected value (fair entry fee).) (`David Rose`)

14. If time allows, we will discuss unsolved problems from previous sets: (5-8 (Shelley), 5-9 (Richard), 6-5 (Richard), 6-7 (Ben), 6-9 (Lei), 7-8 (Richard))

More Problems:

1. Find a combinatorial explanation for the algebraic identity:

$$\binom{\binom{n/2}{2}}{2} = 3\binom{n+1}{4}.$$

(This is part of the background of the Stanford Math Circle website).

2. (Putnam 1987-B2) Let r, s and t be integers with $0 \leq r$, $0 \leq s$ and $r + s \leq t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

3. Three points are randomly chosen on the unit circle. Find the probability that the origin lies inside the triangle formed by the three points.

4. Choose points A,B,C,D independently and (uniformly) randomly on a unit sphere, what is the probability that short-arc AB intersects short-arc CD ?

5. (Putnam 2004-A5) An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.