Problem Set 8

Discussion: Oct. 25, Oct. 27 (on polynomials and floor functions) The name after the problem is the designated writer of the solution of that problem. (Beth, Nicholas, and Frank are exempted this week)

Discussion Problems

1. (a) Factor the polynomial x^8+98x^4+1 into two factors with integer (not necessarily real) coefficients.
   (b) Find the remainder on dividing $x^{100}-2x^{51}+1$ by x^2-1. (`Shelley`)
   (Hint: (a) $98=100-2$; (b) Bezout's theorem)

2. If x_1 and x_2 are the zeros of the polynomial x^2-6x+1, then for every nonnegative integer n, $x_1^n+x_2^n$ is an integer and not divisible by 5. (`Derek`) (Hint: how about induction?)

3. (VA 1982) Let $p(x)$ be a polynomial of the form $p(x)=ax^2+bx+c$, where a, b and c are integers, with the property that $1<p(1)<p(p(1))<p(p(p(1)))$. Show that $a\geq 0$. (`Brett`) (Hint: by contradiction)

4. (VA 1987) A sequence of polynomials is given by $p_n(x)=a_{n+2}x^2+a_{n+1}x-a_n$, for $n\geq 0$, where $a_0=a_1=1$ and, for $n\geq 0$, $a_{n+2}=a_{n+1}+a_n$. Denote by r_n and s_n the roots of $p_n(x)=0$, with $r_n\leq s_n$. Find $\lim_{n\to\infty} r_n$ and $\lim_{n\to\infty} s_n$. (`Ben`) (Hint: think about r_n+s_n and r_ns_n.)

5. (VA 1991) Prove that if α is a real root of $(1-x^2)(1+x+x^2+\cdots+x^n)-x=0$ which lies in $(0,1)$, with $n=1,2,\cdots$, then α is also a root of $(1-x^2)(1+x+x^2+\cdots+x^{n+1})-1=0$. (`Lei`) (Hint: use $1+x+x^2+\cdots+x^n=(1-x^{n+1})/(1-x)$.)

6. (VA 1996) Let a_i, $i=1,2,3,4$, be real numbers such that $a_1+a_2+a_3+a_4=0$. Show that for arbitrary real numbers b_i, $i=1,2,3$, the equation $a_1+b_1x+3a_2x^2+b_2x^3+5a_3x^4+b_3x^5+7a_4x^6=0$ has at least one real root which is on the interval $-1\leq x\leq 1$. (`Tina`) (Hint: think integral)

7. (VA 1995) Let $\tau=(1+\sqrt{5})/2$. Show that $[\tau^2n]=[\tau[\tau n]+1]$ for every positive integer n. Here $[r]$ denotes the largest integer that is not larger than r. (`David Rose`) (Hint: prove $\geq$ and $\leq$ both hold.)

8. Solve the equation $z^8+4z^6-10z^4+4z^2+1=0$. (`Lei`) (Hint: divide it by z^4, and observe the symmetry)

9. (Putnam 2004-B1) Let $P(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r)=0$. Show that the n numbers

$$c_nr,\ c_nr^2+c_{n-1}r,\ c_nr^3+c_{n-1}r^2+c_{n-2}r,$$
$$\ldots,\ c_nr^n+c_{n-1}r^{n-1}+\cdots+c_1r$$

   are integers. (`Davis Edmonson`)

10. (Putnam 2003-B1) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically? (`Richard`)

More Problems:

1. If a and b are two solutions of $x^4 - x^3 - 1 = 0$, then ab is a solution of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

2. Suppose that a, b, c are distinctive integers. Prove

$$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-a)(x-c)}{(b-a)(b-c)} + \frac{c^2(x-b)(x-a)}{(c-b)(c-a)} = x^2$$

for any $x \in \mathbf{R}$.

3. (VA 1997) Suppose that $r1 \neq r2$ and $r1 \cdot r2 = 2$. If $r1$ and $r2$ are roots of $x^4 - x^3 + ax^2 - 8x - 8 = 0$, find $r1$, $r2$ and a. (Do not assume that they are real numbers.)

4. (VA 1991) Let $f(x) = x^5 - 5x^3 + 4x$. In each part (i)–(iv), prove or disprove that there exists a real number c for which $f(x) - c = 0$ has a root of multiplicity (i) one, (ii) two, (iii) three, (iv) four.

5. (VA 1985) Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$, where the coefficients a_i are real. Prove that $p(x) = 0$ has at least one root in the interval $0 \leq x \leq 1$ if $a_0 + a_1/2 + \cdots + a_n/(n+1) = 0$.

6. (VA 1989) Let a,b, c,d be distinct integers such that the equation $(x-a)(x-b)(x-c)(x-d) - 9 = 0$ has an integer root r. Show that $4r = a+b+c+d$. (This is essentially a problem from the 1947 Putnam examination.)

7. (VA 1988) Find positive real numbers a and b such that $f(x) = ax - bx^3$ has four extrema on $[-1, 1]$, at each of which $|f(x)| = 1$.

8. (VA 1987) Let $p(x)$ be given by $p(x) = a_0 + a_1x + a_2x^2 + \; + a_nx^n$ and let $|p(x)| \leq |x|$ on $[-1, 1]$. (a) Evaluate a_0. (b) Prove that $|a_1| \leq 1$.

9. (VA 1990) Suppose that $P(x)$ is a polynomial of degree 3 with integer coefficients and that $P(1) = 0$, $P(2) = 0$. Prove that at least one of its four coefficients is equal to or less than -2.

10. (Putnam 2004-A4) Show that for any positive integer n, there is an integer N such that the product $x_1x_2 \cdots x_n$ can be expressed identically in the form

$$x_1x_2 \cdots x_n = \sum_{i=1}^{N} c_i(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1, 0, 1$.

11. (Putnam 2003-A4) Suppose that a, b, c, A, B, C are real numbers, $a \neq 0$ and $A \neq 0$, such that

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$

for all real numbers x. Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC|.$$

12. (Putnam 2003-B1) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

13. (Putnam 2003-B4) Let

$$\begin{aligned} f(z) &= az^4 + bz^3 + cz^2 + dz + e \\ &= a(z - r_1)(z - r_2)(z - r_3)(z - r_4) \end{aligned}$$

where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.