Problem Set 6

Discussion: Oct. 4, Oct. 6 (mostly on congruence or number theory) The name after the problem is the designated writer of the solution of that problem. (Brett, Shelley, and Frank are exempted this week)

Discussion Problems

- 1. (VT 1983) Let f(x) = 1/x and g(x) = 1 x for $x \in (0, 1)$. List all distinct functions that can be written in the form $f \circ g \circ f \circ g \circ \cdots \circ f \circ g \circ f$ where \circ represents composition. Write each function in the form (ax+b)/(cx+d), and prove that your list is exhaustive. (Tina) (Hint: use matrix)
- 2. (a) (UIUC 2003 Mock) Let f(x) = 1/(1-x). Let f₁(x) = f(x) and for each n = 2, 3, ..., let f_n(x) = f(f_{n-1}(x)). What is the value of f₂₀₀₃(2003)?
 (b) (UIUC 1997) Let x₁ = x₂ = 1, and x_{n+1} = 1996x_n + 1997x_{n-1} for n ≥ 2. Find (with proof) the reminder of x₁₉₉₇ upon division by 3. (Nicholas) (Hint: (a) find the pattern; (b) find the periodic pattern modulo 3)
- 3. (UIUC 1997) Let $x_0 = 0$, $x_1 = 1$, and $x_{n+1} = \frac{x_n + nx_{n-1}}{n+1}$ for $n \ge 1$. Show that the sequence $\{x_n\}$ converges and finds its limit. (Beth) (Hint: guess a general formula)
- 4. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 \neq 0$ and $a_{n+3} = 2a_{n+2} + 5a_{n+1} 6a_n$. Find all possible values for $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$. (David Edmonson) (Hint: find general solution, this is linear)
- 5. The sequence x_n is defined by

$$x_1 = 2, \ x_{n+1} = \frac{2 + x_n}{1 - 2x_n},$$

Prove that (a) $x_n \neq 0$ for all n; (b) x_n is not periodic. (Richard) (Hint: use matrix)

- 6. $a_1 = 0, a_{n+1} = 5a_n + \sqrt{24a_n^2 + 1}$. Prove that a_n is an integer for all n. (David Rose) (Hint: prove it is 2nd order linear)
- 7. $a_0 = 1, a_1 = 5, a_n = \frac{2a_{n-1}^2 3a_{n-1} 9}{2a_{n-2}}$. Prove that a_n is an integer for all n. (Ben) (Hint: similar to No. 10: prove it is a 2nd order linear)
- 8. (Putnam 1980-B3) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n n^2$ have $u_n > 0$ for all $n \ge 0$? (express the answer in the simplest form) (Derek) (Hint: solve the recurrence)
- 9. (Putnam 1956-B6) $a_1 = 2$, $a_{n+1} = a_n^2 a_n + 1$. (a) Prove that any two terms in $\{a_n\}$ are relatively prime; (b) Prove that $\sum_{n=1}^{\infty} 1/a_n = 1$. (Lei) (Hint: $b_n = a_n 1$)
- 10. (Putnam 1993-A2) Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$ Prove there exists a real number a such that $x_{n+1} = ax_n x_{n-1}$ for all $n \geq 1$. (Erin) (Hint: prove $(x_{n+1} + x_{n-1})/x_n$ is a constant)

More Problems:

- 1. (Putnam 1970-A4) Given a sequence $\{x_n\}, n = 1, 2, \cdots$, such that $\lim_{n \to \infty} (x_n x_{n-2}) = 0$. 0. Prove that $\lim_{n \to \infty} \frac{x_n - x_{n-1}}{n} = 0$.
- 2. (Putnam 1966-A3) Let $0 < x_0 < 1$, and $x_{n+1} = x_n(1-x_n)$ for $n \ge 0$. Prove that the limit $\lim_{n \to \infty} nx_n$ exists and is equal to 1.
- 3. (Putnam 1969-B3) The terms of a sequence T_n satisfy $T_n T_{n+1} = n$ $(n = 1, 2, 3, \cdots)$ and $\lim_{n \to \infty} \frac{T_n}{T_{n+1}} = 1$. Show that $\pi T_1^2 = 2$.
- 4. (UIUC 1999) Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$, and $x_{n+1} = \sqrt{2}^{x_n}$ for $n \ge 1$. Prove the sequence $\{x_n\}$ converges and find its limit.
- 5. (Putnam 1994-A1) Suppose that a sequence a_1, a_2, a_3, \ldots satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.
- 6. (Putnam 1997-A6) For a positive integer n and any real number c, define x_k recursively by $x_0 = 0, x_1 = 1$, and for $k \ge 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \le k \le n$.

7. (Putnam 1999-A6) The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$$

Show that, for all n, a_n is an integer multiple of n.

- 8. (UIUC 1995) Let c be a positive constant, let $0 < x_1 < x_0 < 1$, and for $n \ge 1$ let $x_{n+1} = cx_n x_{n-1}$. Prove that there exists a positive real number α such that the limit $L = \lim_{n \to \infty} \frac{x_{n+1}}{x_n^{\alpha}}$ exists and $0 < L < \infty$.
- 9. (Putnam 1985-A3) Let d be a real number. For each integer $m \ge 0$, define a sequence $\{a_m(j)\}, j = 0, 1, 2, \dots$ by the condition

$$a_m(0) = d/2^m,$$

 $a_m(j+1) = (a_m(j))^2 + 2a_m(j), \qquad j \ge 0.$

Evaluate $\lim_{n\to\infty} a_n(n)$.

10. (Putnam 1958-A2) Define
$$a_1 = 1$$
, $a_{n+1} = 1 + \frac{n}{a_n}$. Show that $\sqrt{n} \le a_n < 1 + \sqrt{n}$.