Problem Set 5

Discussion: Sept. 27, Sept. 29 (mostly on congruence or number theory) The name after the problem is the designated writer of the solution of that problem. (Derek, David Edmonson, and Nicholas are exempted this week)

Discussion Problems

1. (a) Four-digit number $S = aabb$ is a square. Find it; (*hint: 11 is a factor of S*)
   (b) If n is a sum of two square, so is $2n$. (*hint: simple algebra*) (`Frank`)

2. (a) If n is an even number, then $323|20^n + 16^n - 3^n - 1$; (*hint: factorize* 323)
   (b) If n is an integer, then $9|4^n + 15n - 1$. (*hint: consider cases when n modulus* 3) (`Ben`)

3. (a) If $2n+1$ and $3n+1$ are squares, then $5n+3$ is not a prime; (*hint: express* $5n+3$ *by* $2n+1$ *and* $3n+1$)
   (b) If $3n+1$ and $4n+1$ are squares, then $56|n$. (*hint: follow the idea in presentation problem*) (`Beth`)

4. (a) If p is a prime, then $p^2 \equiv 1(mod24)$; (*hint: prove* $24|p^2 - 1$)
   (b) Show that if n divides a single Fibonacci number, then it will divide infinitely many Fibonacci numbers. (*hint: think Problem Set 2 number 10.*) (`Tina`)

5. (a) (VT 1979) Show that for all positive integers n, that 14 divides $3^{4n+2} + 5^{2n+1}$;
   (b) (VT 1981) $2^{48} - 1$ is exactly divisible by what two numbers between 60 and 70? (*hint: (a)* $14 = 2 \cdot 7$, *(b) factorizing*) (`Lei`)

6. (a) (VT 1982) What is the remainder when $X^{1982} + 1$ is divided by $X - 1$? Verify your answer (*hint: too simple*);
   (b) (MIT training 2 star) Let n be an integer greater than one. Show that $n^4 + 4^n$ is not prime. (*hint: there is a magic identity due to Sophie Germain:* $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$) (`Erin`)

7. (VT 1988) Let a be a positive integer. Find all positive integers n such that $b = a^n$ satisfying the condition that a^2+b^2 is divisible by $ab+1$. (*hint: prove that* $a^m+1|a^n+1$, *then* $m|n$.) (`Brett`)

8. (Putnam 1972-A5) Show that if n is an integer greater than 1, then n does not divide $2^n - 1$. (`Shelley`)

9. (Putnam 1986-A2) What is the units (*i.e.*, rightmost) digit of $\left[\dfrac{10^{20000}}{10^{100} + 3}\right]$? Here $[x]$ is the greatest integer $\leq x$. (`Richard`)

10. (Putnam 1998-A4) Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n. (`David Rose`)

    Hints for 8-10, see Putnam problem answer online

More Problems:

1. (Putnam 1998-B6) Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

2. (Putnam 1985-A4) Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

3. (Putnam 1955-B4) Do there exist 1,000,000 consecutive integers each of which contains a repeated prime factor?

4. (Putnam 1956-A2) Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

5. (Putnam 1966-B2) Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

6. (MIT training 2.5 star) Let d be any divisor of an integer of the form $n^2 + 1$. Prove that $d - 3$ is not divisible by 4.

7. (MIT training 3 star) What is the last nonzero digit of 10000!?

8. (MIT training 2.5 star) Let n be an integer, and suppose that $n^4 + n^3 + n^2 + n + 1$ is divisible by k. Show that either k or $k - 1$ is divisible by 5.