Problem Set 4

Discussion: Sept. 21, Sept. 23 (mostly on sums) The name after the problem is the designated writer of the solution of that problem. (Brett, Beth and Lei are exempted this week)

Discussion Problems

1. (UIUC 2003) Let $N = 9 + 99 + 999 + \cdots + \overbrace{99\cdots9}^{99}$. Determine the sum of digits of N. `(Derek)`

2. (UIUC 1998) Evaluate $\sum_{k=1}^{n} \frac{k}{2^{k-1}}$. `(David Edmonson)`

3. (UIUC 2004) Let F_n denote the Fibonacci sequence, defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Evaluate $\sum_{n=1}^{\infty} \frac{F_n}{3^n}$. `(Nicholas)`

4. (VT 2003) Evaluate $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots$. `(Ben)`

5. (Putnam 1977 B-1) Evaluate the infinite product: $\prod_{n=1}^{\infty} \frac{n^3-1}{n^3+1}$. `(David Rose)`

6. (Putnam 1978 B-2) Express $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^2n + mn^2 + 2mn}$ as a rational number. `(Shelley)`

7. (Putnam 1977 A-4) For $0 < x < 1$, express $\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$ as a rational function of x.

   (UIUC 2000) Evaluate $\frac{1}{2^1 - 2^{-1}} + \frac{1}{2^2 - 2^{-2}} + \frac{1}{2^4 - 2^{-4}} + \frac{1}{2^8 - 2^{-8}} + \cdots$ `(Tina)`

8. (Putnam 1984 A-2) Express $\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$ as a rational number. `(Erin)`

9. (Putnam 1997 A-3) Evaluate

$$\int_0^{\infty} \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) dx$$

   `(Richard)`

10. (Putnam 1999 A-4) Sum the series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)}$ `(Frank)`

More Problems:

1. (Putnam 1996 B-2) Show that for every positive integer n,
$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}$$

2. (Putnam 1986 A-3) Evaluate $\displaystyle\sum_{n=0}^{\infty} \cot^{-1}(n^2+n+1)$

3. (Putnam 2001 B-3) For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate
$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

4. (Putnam 2004 B-5) Evaluate
$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$