Problem Set 3

Discussion: Sept. 14, Sept. 16 (mostly on mathematical induction) The name after the problem is the designated writer of the solution of that problem. (Erin, Ben and Tina are exempted this week)

Discussion Problems

1. Prove that $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ is an integer for $n = 0, 1, \cdots,$. `(Shelley)`
2. Show that, if n is odd, then $1^n + 2^n + \cdots + n^n$ is divisible by n^2. `(Lei)`
3. Let r be a number such that $r + 1/r$ is an integer. Prove that for every positive integer n, $r^n + 1/r^n$ is an integer. `(Richard)`
4. We need to put n cents of stamps on an envelop, but we have only (an unlimited supply of) 5 cents and 12 cents stamps. Prove that we can perform the task if $n \geq 44$. `(Beth)`
5. Let n be a positive integer. Which one is larger: n^{n+1} or $(n+1)^n$? `(David Edmonson)`
6. Show that for all n, $2^{3^n} + 1$ is divisible by 3^{n+1}. `(Brett)`
7. Given a $(2m + 1) \times (2n + 1)$ checkerboard where the four corner squares are black, show that if one removes any one red and two black squares, the remaining board can be covered with dominoes (1×2 rectangles). `(Frank)`
8. A group of n people play a round-robin tournament. Each game ends in either a win or a loss. Show that it is possible to label the players P_1, P_2, P_3, ..., P_n in such a way that P_1 defeated P_2, P_2 defeated P_3, ... , P_{n-1} defeated P_n. `(Nicholas)`
9. (Putnam 2004 A3) Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that
$$det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$
for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.) `(David Rose)`
10. If each person, in a group of n people, is a friend of at least half the people in the group, then it is possible to seat the n people in a circle so that everyone sits next to friends only. `(Derek)`

Challenging Problems:

1. Show that among any 6 points in a 3×4 rectangle there is a pair of points not more than $\sqrt{5}$ apart.
2. (IMO 2005) In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $2/5$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.