Problem Set 2

Discussion: Sept. 7, Sept. 9 (mostly on pigeon hole principle) The name after the problem is the designated writer of the solution of that problem. (Frank, Nicholas, Derek and Tina are exempted this week)

Discussion Problems

1. (Putnam, 1978-A1) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \cdots, 100$. Prove that there must be two distinct integers in A whose sum is 104. [Actually, 20 can be replaced by 19.] `(Meghan)`

2. (Putnam, 2002-A2) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere. `(Ben)`

3. 15 people sit around a table. When they sit down, they did not notice that a name tag is in front of each seat, and they found that no any name tag and the person sitting there match each other. Prove that after some rotation of the sitting order, at least two people will match the name tag where they sit. `(Brett)`

4. Given any $n+1$ integers between 1 and $2n$, show that one of them is divisible by another. Is this best possible, *i.e.*, is the conclusion still true for n integers between 1 and $2n$? `(Brett)`

5. Six circles with radius 1 is randomly put inside of a circle with radius 6. Prove that at least one more circle with radius 1 can be put inside the big circle without intersecting the other six. `(Beth)`

6. A city has 10000 different telephone lines numbered by 4-digit numbers. More than half of the telephone lines are in the downtown. Prove that there are two telephone numbers in the downtown whose sum is again the number of a downtown telephone line. `(Erin)`

7. Suppose a musical group has 11 weeks to prepare for opening night, and they intend to have at least one rehearsal each day. However, they decide not to schedule more than 12 rehearsals in any 7-day period, to keep from getting burned out. Prove that there exists a sequence of successive days during which the band has exactly 21 rehearsals. `(Shelley)`

8. Prove that there exists a multiple of 2005 whose decimal expansion contains only digits 1 and 0. `(Richard)`

9. (UIUC 2000) Suppose that $a_1, a_2, \cdots, a_n$ are n given integers. Prove that there exist integers r and s with $0 \le r < s \le n$ such that $a_{r+1} + a_{r+2} + \cdots + a_s$ is divisible by n. `(Richard)`

10. The Fibonacci sequence is defined by $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Prove that for any integer m, there exists a_k such that a_k ends with m zeros. `(David Edmonson)`

11. (a) A small party has six people. Each two people either know or dont know each other. Prove there are 3 people in the party such that either they all know each other, or nobody knows each other. (b) (VA Tech 2004-6) An enormous party has an infinite number of people. Each two people either know or dont know each other. Given a positive integer n, prove there are n people in the party such that either they all know each other, or nobody knows each other (so the first possibility means that if A and B are any two of the n people, then A knows B, whereas the second possibility means that if A and B are any two of the n people, then A does not know B). `(Lei)`

12. (Larson page 29 1.5.7) `(David Rose)`

Challenging Problems:

1. Given any $2n$ integers, show that there are n of them whose sum is divisible by n. (Though superficially similar to some other pigeonhole problems, this problem is much more difficult and does not really involve the pigeonhole principle.)

2. (a) Show that among any 9 points in a triangle of area 1, there are 3 points that form a triangle of area at most $1/4$. (b) Show that given any 9 points in a triangle of area 1, there is a triangle of area at least $1/12$ that does not contain any of those 9 points in its interior. (Can you improve $1/12$?)

3. Im going to give each of you a hat to wear that is either black or white. You cannot see a hat on your head, but you can see a hat on anyone elses head. Your Goal as a group is to organize yourselves in a line, with all the white-hatted folks on one end, and all the black-hatted folks on the other end. Note that after I give you your hats, you cannot communicate with (nor touch) anyone. However, before I give you the hats you may jointly decide on a strategy. Devise a strategy that will achieve the Goal.

Presentation discussion problem: (Mathematical Induction)

1. (Putnam 2002 B-1) You have coins $C1, C2, \ldots, Cn$. For each k, Ck is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

2. Show that for $n \geq 6$ a square can be dissected into n smaller squares, not necessarily all of the same size.