Problem Set 11 (last one!)

Discussion: Nov. 22, Nov. 29 (on linear algebra) The name after the problem is the designated writer of the solution of that problem. (Lei, Brett, and Richard are exempted this week)

Discussion Problems

1. (VT 1981) Let A be non-zero square matrix with the property that $A^3 = 0$, where 0 is the zero matrix, but with A being otherwise arbitrary. (a) Express $(I - A)^{-1}$ as a polynomial in A, where I is the identity matrix. (b) Find a 3×3 matrix satisfying $B^2 \neq 0$, $B^3 = 0$. (`Nicholas`)

2. (VT 1979) Let A be an $n \times n$ nonsingular matrix with complex elements, and let $\overline{A}$ be its complex conjugate. Let $B = A\overline{A} + I$, where I is the $n \times n$ identity matrix. (a) Prove or disprove: $A^{-1}BA = \overline{B}$. (b) Prove or disprove: the determinant of $A\overline{A} + I$ is real. (`Tina`)

3. (VT 2003) Determine all invertible 2 by 2 matrices A with complex numbers as entries satisfying $A = A^{-1} = A'$, where A' denotes the transpose of A. (`Beth`)

4. (VT 2002) Let S be a set of 2×2 matrices with complex numbers as entries, and let T be the subset of S consisting of matrices whose eigenvalues are ± 1 (so the eigenvalues for each matrix in T are $\{1, 1\}$ or $\{1, -1\}$ or $\{-1, -1\}$). Suppose there are exactly three matrices in T. Prove that there are matrices A, B in S such that AB is not a matrix in S ($A = B$ is allowed). (`David Edmonson`)

5. (Putnam 1990-A5) If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$? (`Shelley`)

6. (Putnam 1969-B6) Let A and B be matrices of size 3×2 and 2×3 respectively. Suppose that the their product in the order AB is given by

$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}.$$

   Show that the product BA is given by

$$BA = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

   (`Erin`)

7. (Putnam 1994-A4) Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries. (`Ben`)

8. (Putnam 1996-B4) For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

(`David Rose`)

9. (Putnam 1992-B5) Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}.$$

Is the set $\left\{\frac{D_n}{n!}\right\}_{n \geq 2}$ bounded? (`Derek`)

10. (Putnam 1981-B4) A is a set of 5×7 real matrices closed under scalar multiplication and addition. It contains matrices of ranks $0, 1, 2, 4$ and 5. Does it necessarily contain a matrix of rank 3? (`Frank`)