Problem Set 10

Discussion: Nov. 15, Oct. 17 (on games, strategy and puzzle) The name after the problem is the designated writer of the solution of that problem. (Ben, David Edmonson, and Shelley are exempted this week)

Discussion Problems

1. A and B start with $p = 1$. Then they alternately multiply p by one of the numbers 2 to 9. The winner is the one who first reaches (a) $p \geq 1000$, (b) $p \geq 10^6$. Who wins, A or B? (`Derek`)

2. (Putnam 1993-B2) Consider the following game played with a deck of 2n cards numbered from 1 to 2n. The deck is randomly shuffled and n cards are dealt to each of two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by 2n+1. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins? (`Brett`)

3. (Putnam 1995-B5) A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking either (a) one bean from a heap, provided at least two beans are left behind in that heap, or (b) a complete heap of two or three beans. The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy. (`Nicholas`)

4. (Putnam 2002-B2) Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

   > Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

   Show that the player who signs first will always win by playing as well as possible. (`Beth`)

5. (Putnam 2002-B4) An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than 2/3. (`Lei`)

6. (SUMS 2003) Oscar and Nicole are playing a game with matchsticks: They form two piles of matches, one with 42 matches, and the other with 86. They take turns removing matches from the piles, according to the following rules: at each stage the matches taken (at least 1) must all come from one pile, and the number taken must be a divisor of the number of matches in the other pile. The player who removes the last match wins. Nicole goes first. Describe a strategy for Oscar to adopt so that he wins the game, no matter what Nicole does. Show that if we instead start with piles

of 40 and 86 matches, then Nicole can always win, if she adopts the correct strategy. (`Frank`)

7. (Internet math puzzle) Sally and Sue have a strong desire to date Sam. They all live on the same street yet neither Sally or Sue know where Sam lives. The houses on this street are numbered 1 to 99.

   Sally asks Sam "Is your house number a perfect square?". He answers. Then Sally asks "Is is greater than 50?". He answers again.

   Sally thinks she now knows the address of Sam's house and decides to visit.

   When she gets there, she finds out she is wrong. This is not surprising, considering Sam answered only the second question truthfully.

   Sue, unaware of Sally's conversation, asks Sam two questions. Sue asks "Is your house number a perfect cube?". He answers. She then asks "Is it greater than 25?". He answers again.

   Sue thinks she knows where Sam lives and decides to pay him a visit. She too is mistaken as Sam once again answered only the second question truthfully.

   If I tell you that Sam's number is less than Sue's or Sally's, and that the sum of their numbers is a perfect square multiplied by two, you should be able to figure out where all three of them live. (`Tina and Erin`)

8. (Internet math puzzle) Mr. S. and Mr. P. are both perfect logicians, being able to correctly deduce any truth from any set of axioms. Two integers (not necessarily unique) are somehow chosen such that each is within some specified range. Mr. S. is given the sum of these two integers; Mr. P. is given the product of these two integers. After receiving these numbers, the two logicians do not have any communication at all except the following dialogue:

   (a) Mr. P.: I do not know the two numbers.
   (b) Mr. S.: I knew that you didn't know the two numbers.
   (c) Mr. P.: Now I know the two numbers.
   (d) Mr. S.: Now I know the two numbers.

   Given that the above statements are absolutely truthful, what are the two numbers? (`Davis Rose and Richard`)

More Problems:

1. (VT 1987) On Halloween, a black cat and a witch encounter each other near a large mirror positioned along the y-axis. The witch is invisible except by reflection in the mirror. At $t = 0$, the cat is at $(10, 10)$ and the witch is at $(10, 0)$. For $t > 0$, the witch moves toward the cat at a speed numerically equal to their distance of separation and the cat moves toward the apparent position of the witch, as seen by reflection, at a speed numerically equal to their reflected distance of separation. Denote by (u(t), v(t)) the position of the cat and by (x(t), y(t)) the position of the witch. (a) Set up the equations of motion of the cat and the witch for $t \geq 0$. (b) Solve for x(t) and u(t) and find the time when the cat strikes the mirror. (Recall that the mirror is a

perpendicular bisector of the line joining an object with its apparent position as seen by reflection.)

2. (VT 1997) A business man works in New York and Los Angeles. If he is in New York, each day he has four options; to remain in New York, or to fly to Los Angeles by either the 8:00 a.m., 1:00 p.m. or 6:00 p.m. flight. On the other hand if he is in Los Angeles, he has only two options; to remain in Los Angeles, or to fly to New York by the 8:00 a.m. flight. In a 100 day period he has to be in New York both at the beginning of the first day of the period, and at the end of the last day of the period. How many different possible itineraries does the business man have for the 100 day period (for example if it was for a 2 day period rather than a 100 day period, the answer would be 4)?

3. (VT 1991) A and B play the following money game, where a_n and b_n denote the amount of holdings of A and B, respectively, after the n-th round. At each round a player pays one-half his holdings to the bank, then receives one dollar from the bank if the other player had less than c dollars at the end of the previous round. If $a_0 = .5$ and $b_0 = 0$, describe the behavior of a_n and b_n when n is large, for (i) $c = 1.24$ and (ii) $c = 1.26$.