Problem Set 1

Discussion: August 30, Sept. 2. The name after the problem is the designated writer of the solution of that problem. (Richard, David Rose, Ben and Meghan are exempted this week)

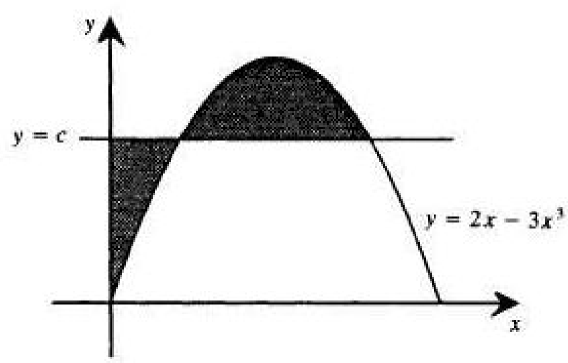
Discussion Problems

1. (VT, 1992-2) Assume that $x1 > y1 > 0$ and $y2 > x2 > 0$. Find a formula for the shortest length l of a planar path that goes from $(x1, y1)$ to $(x2, y2)$ and that touches both the x-axis and the y-axis. Justify your answer. `(Brett)`

2. (VT, 1984-2) Consider any three consecutive positive integers. Prove that the cube of the largest cannot be the sum of the cubes of the other two. `(Beth)`

3. (VT, 1989-2) Let A be a 3×3 matrix in which each element is either 0 or 1 but is otherwise arbitrary. (a) Prove that $det(A)$ cannot be 3 or -3. (b) Find all possible values of $det(A)$ and prove your result. `(Erin)`

4. (Putnam, 1993-A1) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the Figure. Find c so that the areas of the two shaded regions are equal. `(Shelley)`



5. (Putnam, 2001-A1) Consider a set S and a binary operation $*$, *i.e.*, for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$. `(Frank)`

6. (Putnam, 2002-B1) Shanille OKeal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots? `(David Edmonson)`

7. (UIUC, 2005-2) Evaluate the integral $I = \int_0^{\pi} \ln(\sin x)dx$. `(Lei)`

8. (MIT training 2004-12, one star) Let p and q be consecutive odd primes (i.e., no prime numbers are between them). Show that $p + q$ is a product of at least three primes. For instance, $23 + 29$ is the product of the three primes 2, 2, and 13. `(Nicholas)`

9. (Berkeley training) Given an infinite number of points in a plane, prove that if all the distances between them are integers, then the points are all on a straight line. `(Derek)`

10. (Northwestern training) Believe it or not the following function is constant in an interval [a, b]. Find that interval and the constant value of the function. `(Tina)`

$$f(x) = \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$$

Challenging Problems:

1. (MIT training 2004-10, three star) Two players A and B play the following game. Fix a positive real number x. A and B each choose the number 1 or 2. A gives B one dollar if the numbers are different. B gives A x dollars times the sum of their numbers. For instance, if A chooses 1 and B chooses 2, then A gives B one dollar and B gives A $3x$ dollars. Both players are playing their best possible strategy. What value of x makes the game fair, i.e., in the long run both players should break even?

2. (MIT training 2004-7, three star) Into how few pieces can an equilateral triangle be cut and reassembled to form a square?

3. (Putnam 2002-B4) An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an odd number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you must guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.