Linear Algebra

Basic knowledge in linear algebra (from Wikipedia, the free encyclopedia and others)

Linear algebra is the branch of mathematics concerned with the study of vectors, vector spaces (or linear spaces), linear transformations, and systems of linear equations. Vector spaces are a central theme in modern mathematics.

Invertibility of a square matrix

1. A non-zero matrix A with n rows and n columns is invertible if there exists a matrix B that satisfies $AB = BA = I$ where I is the identity matrix.

2. A matrix is invertible if and only if its determinant is different from zero.

3. A matrix is invertible if and only if the linear transformation represented by the matrix is an isomorphism (see also invertible matrix for other equivalent statements).

Determinant Suppose $A = (A_{i,j})$ is a square matrix.
If A is a 1-by-1 matrix, then $\det(A) = A_{1,1}$
If A is a 2-by-2 matrix, then $\det(A) = A_{1,1}A_{2,2} - A_{2,1}A_{1,2}$
For a 3-by-3 matrix A, the formula is more complicated:

$$\begin{aligned}\det(A) \quad = \quad & A_{1,1}A_{2,2}A_{3,3} + A_{1,3}A_{2,1}A_{3,2} + A_{1,2}A_{2,3}A_{3,1} \\ & -A_{1,3}A_{2,2}A_{3,1} - A_{1,1}A_{2,3}A_{3,2} - A_{1,2}A_{2,1}A_{3,3}.\end{aligned}$$

For a general n-by-n matrix, It is also possible to expand a determinant along a row or column using Laplace's formula, which is efficient for relatively small matrices. To do this along row i, say, we write

$$\det(A) = \sum_{j=1}^{n} A_{i,j}C_{i,j} = \sum_{j=1}^{n} A_{i,j}(-1)^{i+j}M_{i,j}$$

where the $C_{i,j}$ represent the matrix cofactors, *i.e.* $C_{i,j}$ is $(-1)^{i+j}$ times the minor $M_{i,j}$, which is the determinant of the matrix that results from A by removing the i-th row and the j-th column.

Solvability of linear systems Let A be a square matrix, and let b be a $n \times 1$ matrix (column vector). The following statements are equivalent:

1. $Ax = b$ has a unique solution for any b;

2. $Ax = 0$ only has the zero solution;

3. A is invertible;

4. $\det(A) \neq 0$.

5. $\lambda = 0$ is not a eigenvalue of A.

Eigenvalues and eigenvectors If $Ax = \lambda x$ for some nonzero vector x and an arbitrary scalar λ, then (λ, x) are said to be an eigenpair, λ being an eigenvalue and x being an eigenvector. Eigenvalues of A can be solved from the characteristic polynomial $P(\lambda) = (-1)^n \det(A - \lambda I) = 0$. If $P(\lambda) = \sum_{i=0}^{n} a_i \lambda^i$, then $a_n = 1$, a_{n-1} is the trace of A (the sum of all eigenvalues), and a_0 is the determinant of A (the product o all eigenvalues). The Cayley-Hamilton Theorem states that replacing λ by the matrix A in the characteristic polynomial results in the zero matrix: $p(A) = 0$.

Example on using Cayley–Hamilton Theorem Consider for example the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. The characteristic polynomial is given by $p(\lambda) = \begin{vmatrix} 1-\lambda & -2 \\ -3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - (-2)(-3) = \lambda^2 - 5\lambda - 2$.

The Cayley–Hamilton theorem then claims that $A^2 - 5A - 2I_2 = 0$ which one can quickly verify in this case. As a result of this, the Cayley–Hamilton theorem allows us to calculate powers of matrices more simply than by direct multiplication.

Taking the result above $A^2 - 5A - 2I_2 = 0$, $A^2 = 5A + 2I_2$. Then, for example, to calculate A^4, observe

$$\begin{aligned} A^3 &= (5A + 2I_2)A = 5A^2 + 2A = 5(5A + 2I_2) + 2A = 27A + 10I_2 \\ A^4 &= A^3 A = (27A + 10I_2)A = 27A^2 + 10A = 27(5A + 2I_2) + 10A \\ A^4 &= 145A + 54I_2. \end{aligned}$$

Matrix power series

Matrix power series is defined as $\sum_{n=0}^{\infty} a_n A^n$ for a square matrix A. For example, $exp(A) = \sum_{n=0}^{\infty} A^n/n!$. An interesting and important fact is that the solution of system of linear differential equation $x' = Ax$ is given by $x(t) = exp(tA)x(0)$, where x is n-dimensional column vector, and A is a $n \times n$ square matrix.

Discussion Problems

1. Calculate the determinant of tri-diagonal matrix:

$$\begin{pmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ . & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{pmatrix}$$

2. (VT 1982) Let a, b, and c be vectors such that $\{a, b, c\}$ is linearly dependent. Show that $\begin{vmatrix} a\cdot a & a\cdot b & a\cdot c \\ b\cdot a & b\cdot b & b\cdot c \\ c\cdot a & c\cdot b & c\cdot c \end{vmatrix} = 0$.

3. (VT 2004) Let I denote the 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $M = \begin{pmatrix} I & A \\ B & C \end{pmatrix}$, $N = \begin{pmatrix} I & B \\ A & C \end{pmatrix}$. where A, B, C are arbitrary 2×2 matrices which entries in $\mathbf{R}$, the

real numbers. Thus M and N are 4×4 matrices with entries in $\mathbf{R}$. Is it true that M is invertible (*i.e.* there is a 4×4 matrix X such that $MX = XM =$ the identity matrix) implies N is invertible? Justify your answer.

4. (Putnam 1991-A2) Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

5. Prove or disprove: There is a real $n \times n$ matrix A such that $A^2 + 2A + 5I = 0$ if and only if n is even.

6. (VT 2000) Let n be a positive integer and let A be an $n \times n$ matrix with real numbers as entries. Suppose $4A^4 + I = 0$, where I denotes the identity matrix. Prove that the trace of A (*i.e.* the sum of the entries on the main diagonal) is an integer.

7. (VT 1992) Let $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Find A^{100}. You have to find all four entries.

8. (Putnam 1994-B4) For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where
$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Show that $\lim_{n\to\infty} d_n = \infty$