Pell's equation: $x^2 - Dy^2 = 1$, $x, y \in \mathbf{N}$, and $D \neq m^2$.

Theorem: (a) Pell's equation has at least one solution; (b) If the smallest solution is (x_1, y_1), then all solutions are given by (x_k, y_k) so that $x_k \pm \sqrt{D} y_k = (x_1 + \sqrt{D} y_1)^k$.

Pythagorean's equation: $x^2 + y^2 = z^2$, $x, y, z \in \mathbf{N}$.

Theorem: The solutions of Pythagorean's equation are given by $x = u^2 - v^2$, $y = 2uv$ and $z = u^2 + v^2$, where u, v are positive integers such that $u > v > 0$.

Trigonometry:

Heron-Qin's formula for area of a triangle: Given the lengths of the sides a, b, and c and the semiperimeter $s = (a + b + c)/2$, of a triangle, Heron's formula gives the area Δ of the triangle as $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.
(For a proof, see `http://mathworld.wolfram.com/HeronsFormula.html`)
Note that another formula from Calculus III is $\Delta = |u \times v|/2$, if the triangle is represented by two vectors u and v, and $\times$ is the cross product.

Qin Jiushao (1202-1261) is one of best ancient Chinese mathematicians. He rediscovered Heron's formula (Heron found it 1000 years earlier), and he got the first general version of Chinese Remainder Theorem (Sun Zi, 1400 years ago, had a special example (see handout of congruence; and 500 years later, Euler and Gauss found it again.) See
`http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Qin_Jiushao.html`

Law of cosine: Let a, b, and c be the sides of the triangle and let A, B, and C the respective angles opposite those sides. Then, $c^2 = a^2 + b^2 - 2ab\cos C$.

Law of sine: If the sides of the triangle are a, b and c, and the angles opposite those sides are A, B and C, then the law of sines states $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

Integral inequalities:

(Cauchy-Schwarz inequality): $\left(\int_a^b |u(x)v(x)|dx\right)^2 \leq \int_a^b u^2(x)dx \cdot \int_a^b v^2(x)dx.$

(Hölder inequality):
$$\int_a^b |u(x)v(x)|dx \leq \left(\int_a^b u^p(x)dx\right)^{1/p} \cdot \left(\int_a^b v^q(x)dx\right)^{1/q}, \text{ where } \frac{1}{p} + \frac{1}{q} = 1,\ p, q > 0.$$

Inequality Presentation Problems

1. Prove that for $a, b, c > 0$, $a^2 + b^2 + c^2 \geq ab + bc + ca$.
2. Prove that for $a, b, c > 0$, $(a + b)(b + c)(c + a) \geq 8abc$.
3. Prove that for $a_i > 0$ $(i = 1, 2, \cdots, n)$, $\displaystyle\sum_{i=1}^n a_i \cdot \sum_{i=1}^n \frac{1}{a_i} \geq n^2$. (Cauchy's)

4. Suppose that $a_i > 0$ $(i = 1, 2, \cdots, n)$ and $a_i \neq a_j$ $(i \neq j)$. Prove $\sum_{i=1}^{n} \frac{a_i^2}{i} \geq \sum_{i=1}^{n} a_i$.

   (use Rearrangement inequality)

5. Prove that for $a, b, c > 0$, $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$.

6. Prove that for all n, $\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n}{2}\right)^n$.

7. (Putnam 2004-A2) For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$? (Use Heron's formula, and other solution can be found online.)

8. (Putnam 2004-B2) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

9. (Putnam 2003-A2) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$\begin{aligned} &(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \\ &\leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}. \end{aligned}$$