Discussion Problem for $[x]$

1. $[x+y] \geq [x]+[y]$
2. The prime p divides $n!$ with multiplicity $I = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$
3. $\left[\frac{x}{n}\right] = \left[\frac{[x]}{n}\right]$.

In the following problems, $[x]$ is the greatest integer $\leq x$.

1. In how many zeros does 10000! end?
2. Prove that for any $n \in \mathbf{N}$,

$$[x] + \left[x+\frac{1}{n}\right] + \left[x+\frac{2}{n}\right] + \left[x+\frac{13}{n}\right] + \cdots + \left[x+\frac{n-1}{n}\right] = [nx].$$

3. Let $a_n = [(1+\sqrt{2})^n]$. Prove that a_n is even when n is even, and a_n is odd when n is odd. (Hint: Consider $(1+\sqrt{2})^n + (\sqrt{2}-1)^n$)
4. (Beatty Theorem) Let a and b be irrational numbers such that $\frac{1}{a} + \frac{1}{b} = 1$. Then the sequences $a_n = [na]$ and $b_n = [nb]$, $n = 1, 2, 3, \cdots$ are disjoint and their union is the set of all natural numbers. (Hint: many websites have this proof.)
5. The range of function $f(n) = [n + \sqrt{n} + 1/2]$ $(n = 1, 2, 3, \cdots)$ is all positive integers except the perfect square numbers. (Hint: if m is not in the range, it means that $n + \sqrt{n} + 1/2 < m < m+1 < n+1+\sqrt{n+1} + 1/2$.)
6. (Putnam 1986-A2) What is the units (*i.e.*, rightmost) digit of $\left[\frac{10^{20000}}{10^{100}+3}\right]$? Here $[x]$ is the greatest integer $\leq x$. (`Richard`)