Fall 2005 William and Mary Mathematical Competition 1

Due: Oct. 12 (Wednesday) 5pm to Jones 122 (Prof. Shi). Each problem is 10 points. Solve as many problems as you can, but you should at least (attempt) give solutions of 6 problems. Try not to use any reference books, but if necessary check for any formulas important for you to solve the problems.

1. (a) Suppose that Fibonacci sequence is given by $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$, Prove that $\displaystyle\sum_{k=1}^{n} \frac{a_k}{2^k} < 2$;
   (b) (VT 1981) Define $F(x)$ by $F(x) = \sum_{n=0}^{\infty} F_n x^n$ (wherever the series converges), where F_n is the n-th Fibonacci number defined by $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n > 1$. Find an explicit closed form (without summation) for $F(x)$.

2. Find the sum: $\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2^{-[3m+n+(m+n)^2]}$.

3. Suppose that $p \geq 5$ is a prime, and $2p+1$ is also a prime. Prove that $4p+1$ is not a prime.

4. Randomly select 55 numbers from the set $S = \{1, 2, 3, \cdots, 99, 100\}$. Prove
   (a) There exists two selected numbers whose difference is 10.
   (b) There exists two selected numbers whose difference is 12.
   (c) It is possible that no any two selected numbers whose difference is 11.

5. (VT 1980) Let $*$ denote a binary operation on a set S with the property that $(w * x) * (y * z) = w * z$ for all $w, x, y, z \in S$. Show (a) If $a * b = c$, then $c * c = c$. (b) If $a * b = c$, then $a * x = c * x$ for all $x \in S$.

6. (VT 1989) Three farmers sell chickens at a market. One has 10 chickens, another has 16, and the third has 26. Each farmer sells at least one, but not all, of his chickens before noon, all farmers selling at the same price per chicken. Later in the day each sells his remaining chickens, all again selling at the same reduced price. If each farmer received a total of \$35 from the sale of his chickens, what was the selling price before noon and the selling price after noon?

7. (VT 1991) Prove that if $x > 0$ and $n > 0$, where x is real and n is an integer, then $\displaystyle\frac{x^n}{(x+1)^{n+1}} \leq \frac{n^n}{(n+1)^{n+1}}$.

8. Prove that if we remove any square from a $2^n \times 2^n$ ($n \geq 1$) checker board, we can always cover the remaining part of the board by L-shaped domino (consisting of three squares).

9. Each of ten line segments is longer than 1 but shorter than 55. Prove that you can select three line segments from these ten to form a triangle.

10. For any positive integer n, prove that there exists a multiple of 2^n whose each digit (base 10) is not zero.