

Binomial identities, binomial coefficients, and binomial theorem (from Wikipedia, the free encyclopedia)

In mathematics, the binomial theorem is an important formula giving the expansion of powers of sums. Its simplest version reads

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

whenever n is any non-negative integer, the numbers

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

are the binomial coefficients, and $n!$ denotes the factorial of n .

This formula, and the triangular arrangement of the binomial coefficients, are often attributed to Blaise Pascal who described them in the 17th century. It was, however, known to Chinese mathematician Yang Hui in the 13th century.

For example, here are the cases $n = 2$, $n = 3$ and $n = 4$:

$$\begin{aligned}(x + y)^2 &= x^2 + 2xy + y^2, \\(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3, \\(x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

Isaac Newton generalized the formula to other exponents by considering an infinite series:

$$(x + y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k y^{r-k}$$

where r can be any complex number (in particular r can be any real number, not necessarily positive and not necessarily an integer), and the coefficients are given by

$$\binom{r}{k} = \frac{1}{k!} \prod_{n=0}^{k-1} (r - n) = \frac{r(r-1)(r-2) \cdots (r-(k-1))}{k!}.$$

A particularly handy but non-obvious form holds for the reciprocal power:

$$\frac{1}{(1-x)^r} = \sum_{k=0}^{\infty} \binom{r+k-1}{k} x^k \equiv \sum_{k=0}^{\infty} \binom{r+k-1}{r-1} x^k.$$

Pascal's triangle is a geometric arrangement of the binomial coefficients in a triangle. Pascal's triangle can be constructed using Pascal's rule (or addition formula), which states that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for non-negative integers n and k where $n \geq k$ and with $\binom{n}{0} = \binom{n}{n} = 1$. Another two basic properties are symmetry condition:

$$\binom{n}{k} = \binom{n}{n-k},$$

and

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}, \quad k \neq 0.$$

Generating functions:

A generating function is a formal power series whose coefficients encode information about a sequence a_n that is indexed by the natural numbers. Usually generating function of a sequence a_n is

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n.$$

For example, the generating function of $a_n = 1$ is

$$G(1; x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

and the generating function of $a_n = n^2$ is (see “sums” in week 3 for derivation of the function)

$$G(n^2; x) = \sum_{n=0}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}.$$

Consider the problem of finding a closed formula for the Fibonacci numbers f_n defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. We form the ordinary generating function

$$f(X) = \sum_{n=0}^{\infty} f_n X^n$$

for this sequence. The generating function for the sequence (f_{n-1}) is $X \cdot f(X)$ and that of (f_{n-2}) is $X^2 \cdot f(X)$. From the recurrence relation, we therefore see that the power series $Xf(X) + X^2f(X)$ agrees with $f(X)$ except for the first two coefficients. Taking these into account, we find that $f(X) = Xf(X) + X^2f(X) + X$ (this is the crucial step; recurrence relations can almost always be translated into equations for the generating functions). Solving this equation for $f(X)$, we get

$$f(X) = \frac{X}{1-X-X^2}.$$

Example A: Use combinatoric method to prove the identities:

1. $\sum_{k=0}^n \binom{n}{k} = 2^n.$
2. $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0.$
3. $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$
4. $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}.$

$$5. \sum_{k=0}^n \binom{m}{k} \binom{n}{r-k}.$$

$$6. \sum_{k=0}^n k \binom{n}{k} = n2^{n-1}.$$

Example B:

- (a) How many ways can you choose 12 cans of soup from among five different varieties if the order chosen doesn't matter? (b) Same question, but now the order DOES matter.
- How many ways are there to place an order of 10 donuts if there are 3 varieties to choose from?
- How many ways are there to form a committee of 5 in a group of 20 people?
- (Putnam 1985-A1) Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that
 - $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and
 - $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

- (VA) A box contains marbles, each of which is red, white or blue. The number of blue marbles is at least half the number of white marbles and at most one third the number of red marbles. The number which are white or blue is at least 55. Find the minimum possible number of red marbles.
- (UIUC) Let $f(n)$ denote the number of ordered tuples (r_1, \dots, r_k) of positive integers with $r_1 + \dots + r_k = n$. For example, $f(3) = 4$, since 3 has four representations of this type: $3 = 3, 3 = 1 + 2, 3 = 2 + 1, 3 = 1 + 1 + 1$. Find and prove a general formula for $f(n)$.
- (Putnam 2003-A1) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \dots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \dots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: $4, 2+2, 1+1+2, 1+1+1+1$.
- Suppose that n is an even number, and one can select four *distinctive* numbers $a > b > c > d$ from the set $\{1, 2, \dots, n\}$ so that $a + d = b + c$. Prove the number of different selections is $\frac{n(n-2)(2n-5)}{24}$.
- (Putnam 1999-A3) Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$