Problem Set 8

1. (a) Factor the polynomial $x^8 + 98x^4 + 1$ into two factors with integer (not necessarily real) coefficients.
   (b) Find the remainder on dividing $x^{100} - 2x^{51} + 1$ by $x^2 - 1$. (`Shelley`)

   Solution:

   (a) $x^8 + 98x^4 + 1 \Rightarrow x^8 - 2x^4 + 1 + 2x^4 + 98x^4 \Rightarrow x^8 - 2x^4 + 1 + 100x^4 \Rightarrow (x^4 - 1)^2 - (-100x^4) \Rightarrow (x^4 - 1)^2 - (10x^2 i)^2 = [(x^4 - 1) + (10x^2 i)][(x^4 - 1) - (10x^2 i)]$. Done.

   (b) $(x^{100} - 2x^{51} + 1)/(x^2 - 1)$ What is the remainder?

   $x^2 - 1 = (x + 1)(x - 1)$. So two solutions are $x = \pm 1$. Using Bezout's theorem, we substitute these into the original function (the solution must be of the form bx + c because it must be of a lesser degree than the divisor):

   $f(1) = 0 = bx + c = b + c$

   $f(-1) = 4 = -bx + c = -b + c$

   Solving the system of equations, b=-2, c=2. The remainder is -2x + 2.

2. If x_1 and x_2 are the zeros of the polynomial $x^2 - 6x + 1$, then for every nonnegative integer n, $x_1^n + x_2^n$ is an integer and not divisible by 5. (`Derek`) (Hint: how about induction?)

3. (VA 1982) Let $p(x)$ be a polynomial of the form $p(x) = ax^2 + bx + c$, where a, b and c are integers, with the property that $1 < p(1) < p(p(1)) < p(p(p(1)))$. Show that $a \geq 0$. (`Brett`)

   We don't know how to do this one yet!

4. (VA 1987) A sequence of polynomials is given by $p_n(x) = a_{n+2}x^2 + a_{n+1}x - a_n$, for $n \geq 0$, where $a_0 = a_1 = 1$ and, for $n \geq 0$, $a_{n+2} = a_{n+1} + a_n$. Denote by r_n and s_n the roots of $p_n(x) = 0$, with $r_n \leq s_n$. Find $\lim_{n\to\infty} r_n$ and $\lim_{n\to\infty} s_n$. (`Ben`)

   Solution: Since this is a quadratic equation, we can make use of the quadratic formula to come up with equations for the two roots as follows:

$$\begin{aligned}
x &= \frac{-a_{n+1} \pm \sqrt{a_{n+1}^2 - 4(a_{n+2})(-a_n)}}{2a_{n+2}} \\
&= \frac{-a_{n+1} \pm \sqrt{a_{n+1}^2 + 4(a_{n+2})(a_n)}}{2a_{n+2}} \\
&= \frac{-a_{n+1} \pm \sqrt{a_{n+1}^2 + 4(a_n + a_{n+1})(a_n)}}{2a_{n+2}} \\
&= \frac{-a_{n+1} \pm \sqrt{a_{n+1}^2 + 4a_n a_{n+1} + 4a_n^2}}{2a_{n+2}} \\
&= \frac{-a_{n+1} \pm \sqrt{(2a_n + a_{n+1})^2}}{2a_{n+2}} \\
&= \frac{-a_{n+1} \pm (2a_n + a_{n+1})}{2a_{n+2}}
\end{aligned}$$

where we have used the fact that $a_{n+2} \,=\, a_n + a_{n+1}$.

Thus we have,

$$\begin{aligned} r_n &= \frac{-a_{n+1} - 2a_n - a_{n+1}}{2a_{n+2}} \\ &= \frac{-2(a_n + a_{n+1})}{2(a_n + a_{n+1})} \\ &= -1 \end{aligned}$$

and

$$\lim_{n \to +\infty} r_n \,=\, \lim_{n \to +\infty} -1 \,=\, -1.$$

And for the other root we have

$$\begin{aligned} s_n &= \frac{-a_{n+1} + 2a_n + a_{n+1}}{2a_{n+2}} \\ &= \frac{2a_n}{2(a_n + a_{n+1})} \\ &= \frac{a_n}{a_n + a_{n+1}} \end{aligned}$$

Now, we need a formula for a_n, and we know that

$$a_{n+2} = 1 \cdot a_n + 1 \cdot a_{n+1} \tag{0.1}$$

The solution to equation (1) is given by $a_n = c_1\lambda_1^n + c_2\lambda_2^n$, where λ_1 and λ_2 are roots of the characteristic equation $\lambda^2 - A\lambda - B = 0$, and c_1 and c_2 are to be determined by initial conditions (although we will not need them for this problem). So solving the characteristic for its roots via the quadratic formula, we find that:

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

Thus,

$$a_n = c_1 \left[\frac{1+\sqrt{5}}{2}\right]^n + c_2 \left[\frac{1-\sqrt{5}}{2}\right]^n \tag{0.2}$$

Notice that in equation (2) it is the case that $-1 < \lambda_2 < 0$, and since we are evaluating $\lim_{n \to +\infty}$, we can neglect the $c_2\lambda_2^n$ term.

Thus we have

$$\begin{aligned}
\lim_{n\to+\infty} s_n &= \lim_{n\to+\infty} \frac{a_n}{a_n + a_{n+1}} \\
&= \lim_{n\to+\infty} \frac{c_1\left[\frac{1+\sqrt{5}}{2}\right]^n}{c_1\left[\frac{1+\sqrt{5}}{2}\right]^n + c_1\left[\frac{1+\sqrt{5}}{2}\right]^{n+1}} \\
&= \lim_{n\to+\infty} \frac{\left[\frac{1+\sqrt{5}}{2}\right]^n}{\left[\frac{1+\sqrt{5}}{2}\right]^n\left[1+\left[\frac{1+\sqrt{5}}{2}\right]\right]} \\
&= \lim_{n\to+\infty} \frac{1}{1+\left[\frac{1+\sqrt{5}}{2}\right]} \\
&= \lim_{n\to+\infty} \frac{2}{3+\sqrt{5}} \\
&= \frac{6-2\sqrt{5}}{4}.
\end{aligned}$$

5. (VA 1991) Prove that if α is a real root of $(1-x^2)(1+x+x^2+\cdots+x^n)-x=0$ which lies in $(0,1)$, with $n=1,2,\cdots$, then α is also a root of $(1-x^2)(1+x+x^2+\cdots+x^{n+1})-1=0$. (Lei) (Hint: use $1+x+x^2+\cdots+x^n=(1-x^{n+1})/(1-x)$.)

6. (VA 1996) Let a_i, $i=1,2,3,4$, be real numbers such that $a_1+a_2+a_3+a_4=0$. Show that for arbitrary real numbers b_i, $i=1,2,3$, the equation $a_1+b_1x+3a_2x^2+b_2x^3+5a_3x^4+b_3x^5+7a_4x^6=0$ has at least one real root which is on the interval $-1\le x\le 1$. (Tina)

   Solution:

   Taking the integral of the given equation over the interval $-1\le x\le 1$ gives:

$$\begin{aligned}
\int_{-1}^{1}(a_1+b_1x &+3a_2x^2+b_2x^3+5a_3x^4+b_3x^5+7a_4x^6)dx \\
&= a_1x+\tfrac{b_1}{2}x^2+a_2x^3+\tfrac{b_2}{4}x^4+a_3x^5+\tfrac{b_3}{6}x^6+a_4x^7\Big|_{-1}^{1} \\
&= \left(a_1+\tfrac{b_1}{2}+a_2+\tfrac{b_2}{4}+a_3+\tfrac{b_3}{6}+a_4\right)-\left(-a_1+\tfrac{b_1}{2}-a_2+\tfrac{b_2}{4}-a_3+\tfrac{b_3}{6}-a_4\right) \\
&= 2\left(a_1+a_2+a_3+a_4\right)+\left(\tfrac{b_1}{2}+\tfrac{b_2}{4}+\tfrac{b_3}{6}\right)-\left(\tfrac{b_1}{2}+\tfrac{b_2}{4}+\tfrac{b_3}{6}\right) \\
&= 0
\end{aligned}$$

   And by the Mean Value Theorem, since the integral of the equation is zero, the function must take on the value of zero somewhere on the interval $-1\le x\le 1$.

7. (VA 1995) Let $\tau=(1+\sqrt{5})/2$. Show that $[\tau^2 n]=[\tau[\tau n]+1]$ for every positive integer n. Here $[r]$ denotes the largest integer that is not larger than r. (David Rose)

Solution

$$
\begin{aligned}
[\tau[\tau n]+1] &= [\tau(\tau n-\{\tau n\})+1] \\
&= [\tau^2 n-\tau\{\tau n\}+1] \\
&= [(\tau+1)n-\tau\{\tau n\}+1] \\
&= [\tau n-\tau\{\tau n\}+1]+n \\
&= n+[[\tau n]+\{\tau n\}-\tau\{\tau n\}+1] \\
&= n+[\tau n]+[1-(\tau-1)\{\tau n\}] \\
&= n+[\tau n] \\
&= [(\tau+1)n] \\
&= [\tau^2 n]
\end{aligned}
$$

8. Solve the equation $z^8+4z^6-10z^4+4z^2+1=0$. (`Lei`)

9. (Putnam 2004-B1) Let $P(x)=c_n x^n+c_{n-1}x^{n-1}+\cdots+c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r)=0$. Show that the n numbers

$$
c_n r,\ c_n r^2+c_{n-1}r,\ c_n r^3+c_{n-1}r^2+c_{n-2}r,\\
\ldots,\ c_n r^n+c_{n-1}r^{n-1}+\cdots+c_1 r
$$

are integers. (`David Edmonson`)

Solution. Set $i \in \{0, 1, \ldots, n-1\}$. Now write r as $\frac{u}{v}$, where u and v are coprime. Then $c_n(\frac{u}{v})^n + c_{n-1}(\frac{u}{v})^{n-1} + \ldots + c_0 = 0$, so $c_n u^n + c_{n-1}u^{n-1}v + \cdots + c_0 v^n = 0$, so $c_n u^n + c_{n-1}u^{n-1}v + \ldots + c_{i+1}u^{i+1}v^{n-i-1} = -c_i u^i v^{n-i} - c_{i-1}u^{i-1}v^{n-i+1} - \ldots - c_0 v^n$ is a multiple of v^{n-i}, so $c_n u^{n-i} + c_{n-1}u^{n-i-1}v + \ldots + c_{i+1}uv^{n-i-1}$ is also a multiple of v^{n-i}, since u^i and v^{n-i} are coprime. Thus, $c_n(\frac{u}{v})^{n-i} + c_{n-1}(\frac{u}{v})^{n-1-i} + \ldots + c_{i+1}(\frac{u}{v})$ is an integer, which is the claim that we were asked to show.

10. (Putnam 2003-B1) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1+xy+x^2y^2=a(x)c(y)+b(x)d(y)$$

holds identically? (`Richard`)