Problem Set 7

1. (a) Prove that for $a, b, c > 0$ satisfying $(1+a)(1+b)(1+c) = 8$, then $abc \leq 1$.
   (b) Prove that for $a, b, c > 0$, then $(a^2b+b^2c+c^2a)(a^2c+b^2a+c^2b) \geq 9a^2b^2c^2$. (`Derek`)

2. Given that a, b, c, d, e are real numbers such that $a+b+c+d+e = 8$ and $a^2+b^2+c^2+d^2+e^2 = 16$. Find the maximum value of e. (`Shelley`)

   Solution

$$\begin{aligned}(8-e)^2 &= (a+b+c+d)^2 \\ &= a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd) \\ &\leq 4(a^2+b^2+c^2+d^2)\end{aligned}$$

   $(16-e^2) = a^2+b^2+c^2+d^2$
   $4(a^2+b^2+c^2+d^2) \geq (a+b+c+d)^2$
   $4(16-e^2) \geq (8-e)^2$
   $64-4e^2 \geq 64-16e^2+e^2$
   $5e^2-16e \leq 0$
   $e(5e-16) \leq 0, e \leq \frac{16}{5}$

3. (a) Prove that if a and b are positive numbers such that $a = b = 1$,
   then $\left(a+\dfrac{1}{a}\right)^2 + \left(b+\dfrac{1}{b}\right)^2 \geq \dfrac{25}{2}$.
   (b) Prove that if $a_i > 0$ $(i = 1, 2, \cdots, n)$, and $a_1+a_2+\cdots+a_n = 1$. Prove that $\displaystyle\sum_{i=1}^{n}\left(a_i+\frac{1}{a_i}\right)^2 \geq \frac{(n^2+1)^2}{n}$. (`Beth`)
   (a) We notice that a=1-b. This means that

$$f(b) = (1-b+\frac{1}{1-b})^2+(b+\frac{1}{b})^2 \geq \frac{25}{2}$$

   Using calculus, we can take the derivative, and we find that

$$f'(b) = \frac{-2}{(b-1)^3}+4b-\frac{2}{b^3}-2 = 0.$$

   We can graph this, or solve the derivative set to zero, and we get $b = a = \frac{1}{2}$.
   So, plugging this back into the original equation,

$$(\frac{1}{2}+\frac{1}{\frac{1}{2}})^2+(\frac{1}{2}+\frac{1}{\frac{1}{2}})^2 \geq \frac{25}{2}.$$

   and we get that 12.5=12.5, which satisfies the equation.
   (b)

$$\sum_{i=1}^{n}(a_i^2+2+\frac{1}{a_i^2})$$

$$= \sum_{i=1}^{n}a_i^2+2n+\sum_{i=1}^{n}\frac{1}{a_i^2} \geq \frac{(n^2+1)^2}{n} = n^3+2n+\frac{1}{n}.$$

We can eliminate 2n from both sides, and then we get

$$= \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} \frac{1}{a_i^2} \geq n^3 + \frac{1}{n}.$$

Using the Cauchy inequality, we know that

$$(\sum a_i^2)(\sum b_i^2) \geq (\sum a_i b_i)^2.$$

Here b is just 1. So we see that

$$\sum a_i^2 * \sum 1^2 \geq (\sum a_i)^2 = 1^2$$

So,

$$\sum a_i^2 n \geq 1$$

This leads to the conclusion that

$$\sum a_i^2 \geq \frac{1}{n}.$$

Now we just need to prove that

$$\frac{1}{a_i^2} \geq n^3.$$

Say that $\sum a_i^{-2} = S$, then using the Power Mean inequality with $\alpha \neq 0$, we have $\alpha = -2$.

$$M_{-2} \leq M_1 = \frac{a_1 + a_2 + ... + a_n}{n} = \frac{1}{n}$$

$$M_{-2} = (\frac{\sum a_i^{-2}}{n})^{\frac{-1}{2}}$$

$$= (\frac{S}{n})^{\frac{-1}{2}} \leq \frac{1}{n}$$

$$= (\frac{n}{S})^{\frac{1}{2}} \leq \frac{1}{n}$$

If we square both sides,

$$\frac{n}{S} \leq \frac{1}{n^2}$$

$$n^3 \leq S$$

So we have the original inequality because each individual piece on the left has a lesser or equal counterpart on the right hand side.

4. Let a, b, c denote the lengths of the sides of a triangle.
   Show that $\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2$. (`Nicholas`)

5. (Putnam 1998-B1) Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x>0$. (`Ben`)

Solution. For this problem we will need the following facts:

$$(a^2-b^2)=(a+b)(a-b) \tag{0.1}$$

and

$$\left(x+\frac{1}{x}\right)^3=x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right) \tag{0.2}$$

Now we are given

$$\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)} \tag{0.3}$$

Note that

$$\begin{aligned}
\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2 &= \left[\left(x+\frac{1}{x}\right)^3\right]^2-\left(x^6+2+\frac{1}{x^6}\right)\\
&= \left[\left(x+\frac{1}{x}\right)^3\right]^2-\left(x^3+\frac{1}{x^3}\right)^2\\
&= \left[\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)\right]\left[\left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right)\right]
\end{aligned}$$

where we used fact (1).
Thus we have

$$\begin{aligned}
\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)} &= \frac{\left[\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)\right]\left[\left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right)\right]}{\left[\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)\right]}\\
&= \left[\left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right)\right]\\
&= x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)-x^3-\frac{1}{x^3}\\
&= 3\left(x+\frac{1}{x}\right)
\end{aligned}$$

where we used fact (2).

Now to find the minimum of $f(x)=x+\frac{1}{x}$, we need to take the derivative of $f(x)$ and find it's zeroes.

And

$$f'(x) = 1 - \frac{1}{x^2} = 0$$

so

$$1 - \frac{1}{x^2} = 0 \;\Rightarrow\; x^2 = 1 \;\Rightarrow\; x = \pm 1$$

But $x > 0$, so $x = 1$. This is indeed a minimum since $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$. Thus the minimum value of (3) for $x > 0$ is

$$3\left(1 + \frac{1}{1}\right) = 6.$$

6. (Putnam 1973) On $[0, 1]$, let f have a continuous derivative satisfying $0 < f'(t) \leq 1$. Also suppose that $f(0) = 0$. Prove that $\left[\int_0^1 f(t)dt\right]^2 \geq \int_0^1 [f(t)]^3 dt$. (`Frank`)

7. (Putnam 1962-B5) Show that for $n > 1$, $\frac{3n+1}{2n+2} < \sum_{r=1}^{n} \frac{r^n}{n^n} < 2$. (`David Rose`)

   Solution We first show that

$$\sum_{r=1}^{n} \frac{r^n}{n^n} < 2.$$

   Note this is equivalent to showing that

$$1^n + 2^n + \cdots + (n-1)^n < n^n.$$

   Now, for any $a \leq n$, we have by the binomial expansion

$$a^n = ((a-1)+1)^n = (a-1)^n + n(a-1)^{n-1} + \cdots > (a-1)^n + (a-1)^n.$$

   Thus we have

$$n^n > (n-1)^n + (n-1)^n > (n-1)^n + (n-2)^n + (n-2)^n > \cdots > (n-1)^n + \cdots + 1^n$$

   as desired. We will now show that

$$\frac{3n+1}{2n+2} < \sum_{r=1}^{n} \frac{r^n}{n^n}.$$

   Consider the function $f(x) = x^n$ on the interval $[0, 1]$. We approximate this integral using Riemann sums, and we will subtract the area of the upper triangles, noting that this will still overestimate the integral, since x^n is concave-up on this interval. We thus have the inequality:

$$\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^n - \frac{1}{2n}\sum_{r=1}^{n}\left[\left(\frac{r}{n}\right)^n - \left(\frac{r-1}{n}\right)^n\right] > \int_0^1 x^n dx.$$

   Simplifying, we see that

$$\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^n - \frac{1}{2n} > \frac{1}{n+1}$$

   or simply

$$\sum_{r=1}^{n}\left(\frac{r}{n}\right)^n > \frac{3n+1}{2n+2}$$

8. (Putnam 2003-A3) Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x. (`Richard`)

9. (Putnam 2004-A2) For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$? (Hint: Use Heron's formula, and other solution can be found online.) (`Lei`)

10. (Putnam 2004-B2) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

(`Brett`)