Problem Set 7

1. (a) Prove that for $a, b, c > 0$ satisfying $(1 + a)(1 + b)(1 + c) = 8$, then $abc \leq 1$.
   (b) Prove that for $a, b, c > 0$, then $(a^2 b + b^2 c + c^2 a)(a^2 c+b^2 a + c^2 b) \geq 9a^2 b^2 c^2$. (Derek)

2. Given that $a, b, c, d, e$ are real numbers such that $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$. Find the maximum value of $e$. (Shelley)

Solution

\[
(8 - e)^2 = (a + b + c + d)^2 \\
= a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd) \\
\leq 4(a^2 + b^2 + c^2 + d^2)
\]

\[
16 - e^2 = a^2 + b^2 + c^2 + d^2 \\
4(a^2 + b^2 + c^2 + d^2) \geq (a + b + c + d)^2 \\
4(16 - e^2) \geq (8 - e)^2 \\
64 - 4e^2 \geq 64 - 16e^2 + e^2 \\
5e^2 - 16e \leq 0 \\
e(5e - 16) \leq 0, e \leq \frac{16}{5}
\]

3. (a) Prove that if $a$ and $b$ are positive numbers such that $a = b = 1$,
then \( \left( a + \frac{1}{a} \right)^2 + \left( b + \frac{1}{b} \right)^2 \geq \frac{25}{2}. \)
   (b) Prove that if $a_i > 0$ $(i = 1, 2, \cdots, n)$, and $a_1 + a_2 + \cdots + a_n = 1$. Prove that
   \( \sum_{i=1}^{n} \left( a_i + \frac{1}{a_i} \right)^2 \geq \frac{(n^2 + 1)^2}{n}. \) (Beth)

(a) We notice that a=1-b. This means that
   \[ f(b) = (1 - b + \frac{1}{1-b})^2 + (b + \frac{1}{b})^2 \geq \frac{25}{2} \]

Using calculus, we can take the derivative, and we find that
   \[ f'(b) = \frac{-2}{(b-1)^{3}} + 4b - \frac{2}{b^{2}} - 2 = 0. \]

We can graph this, or solve the derivative set to zero, and we get $b = a = \frac{1}{2}$.

So, plugging this back into the original equation,
   \[ (\frac{1}{2} + \frac{1}{2})^2 + (\frac{1}{2} + \frac{1}{2})^2 \geq \frac{25}{2}. \]

and we get that 12.5=12.5, which satisfies the equation.

(b) \[
\sum_{i=1}^{n} (a_i^2 + 2 + \frac{1}{a_i^2}) \\
= \sum_{i=1}^{n} a_i^2 + 2n + \sum_{i=1}^{n} \frac{1}{a_i^2} \geq \frac{(n^2 + 1)^2}{n} = n^2 + 2n + \frac{1}{n}.
\]
We can eliminate $2n$ from both sides, and then we get

$$= \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} \frac{1}{a_i^2} \geq n^3 + \frac{1}{n}.$$ 

Using the Cauchy inequality, we know that

$$\left(\sum a_i^2\right) \left(\sum b_i^2\right) \geq \left(\sum a_i b_i\right)^2.$$ 

Here $b$ is just 1. So we see that

$$\sum a_i^2 \cdot 1^2 \geq \left(\sum a_i\right)^2 = 1^2$$

So,

$$\sum a_i^2 \geq \frac{1}{n}.$$ 

This leads to the conclusion that

$$\sum a_i^2 \geq n^3.$$ 

Now we just need to prove that

$$\frac{1}{a_i^2} \geq \frac{1}{n}.$$ 

Say that $\sum a_i^{-2} = S$, then using the Power Mean inequality with $\alpha \neq 0$, we have $\alpha = -2$.

$$M_{-2} \leq M_1 = \frac{a_1 + a_2 + \ldots + a_n}{n} = \frac{1}{n}$$

$$M_{-2} = \left(\frac{\sum a_i^{-2}}{n}\right)^{-\frac{1}{2}}$$

$$= \left(\frac{S}{n}\right)^{-\frac{1}{2}} \leq \frac{1}{n}$$

$$= \left(\frac{n}{S}\right)^{\frac{1}{2}} \leq \frac{1}{n}$$

If we square both sides,

$$\frac{n}{S} \leq \frac{1}{n^2}$$

$$n^3 \leq S$$

So we have the original inequality because each individual piece on the left has a lesser or equal counterpart on the right hand side.

4. Let $a, b, c$ denote the lengths of the sides of a triangle.

Show that $\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2$. (Nicholas)
5. (Putnam 1998-B1) Find the minimum value of

\[
\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}
\]

for \(x > 0\). (Ben)

**Solution.** For this problem we will need the following facts:

\[(0.1) \quad (a^2 - b^2) = (a + b)(a - b)\]

and

\[(0.2) \quad \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3 \left(x + \frac{1}{x}\right)\]

Now we are given

\[(0.3) \quad \frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}\]

Note that

\[
\left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2 = \left\{ \left(x + \frac{1}{x}\right)^3 \right\}^2 - \left(x^6 + 2 + \frac{1}{x^6}\right)
\]

\[
= \left\{ \left(x + \frac{1}{x}\right)^3 \right\}^2 - \left(x^3 + \frac{1}{x^3}\right)^2
\]

\[
= \left(x + \frac{1}{x}\right)^3 + \left(x^3 + \frac{1}{x^3}\right) \left(x + \frac{1}{x}\right)^3 - \left(x^3 + \frac{1}{x^3}\right)
\]

where we used fact \((1)\).

Thus we have

\[
\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)} = \frac{(x + \frac{1}{x})^3 + (x^3 + \frac{1}{x^3}) \left[(x + \frac{1}{x})^3 - (x^3 + \frac{1}{x^3})\right]}{\left[(x + \frac{1}{x})^3 + (x^3 + \frac{1}{x^3})\right]}
\]

\[
= \left(x + \frac{1}{x}\right)^3 - \left(x^3 + \frac{1}{x^3}\right)
\]

\[
= x^3 + \frac{1}{x^3} + 3 \left(x + \frac{1}{x}\right) - x^3 - \frac{1}{x^3}
\]

\[
= 3 \left(x + \frac{1}{x}\right)
\]

where we used fact \((2)\).

Now to find the minimum of \(f(x) = x + \frac{1}{x}\), we need to take the derivative of \(f(x)\) and find its zeroes.
And

$$f'(x) = 1 - \frac{1}{x^2} = 0$$

so

$$1 - \frac{1}{x^2} = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

But $x > 0$, so $x = 1$. This is indeed a minimum since $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$. Thus the minimum value of (3) for $x > 0$ is

$$3\left(1 + \frac{1}{1}\right) = 6.$$

6. (Putnam 1973) On $[0, 1]$, let $f$ have a continuous derivative satisfying $0 < f'(t) \leq 1$.

Also suppose that $f(0) = 0$. Prove that

$$\left[\int_0^1 f(t)dt\right]^2 \geq \int_0^1 [f(t)]^3 dt. \text{ (Frank)}$$

7. (Putnam 1962-B5) Show that for $n > 1$, $\frac{3n + 1}{2n + 2} < \sum_{r=1}^{n} \frac{r^n}{n^n} < 2$. (David Rose)

**Solution** We first show that

$$\sum_{r=1}^{n} \frac{r^n}{n^n} < 2.$$

Note this is equivalent to showing that

$$1^n + 2^n + \cdots + (n - 1)^n < n^n.$$

Now, for any $a \leq n$, we have by the binomial expansion

$$a^n = ((a - 1) + 1)^n = (a - 1)^n + n(a - 1)^{n-1} + \cdots + (a - 1)^1 + (a - 1)^0.$$

Thus we have

$$n^n > (n - 1)^n + (n - 1)^n + (n - 2)^n + (n - 2)^n + \cdots + (n - 1)^n + \cdots + 1^n$$

as desired. We will now show that

$$\frac{3n + 1}{2n + 2} < \sum_{r=1}^{n} \frac{r^n}{n^n}.$$

Consider the function $f(x) = x^n$ on the interval $[0, 1]$. We approximate this integral using Riemann sums, and we will subtract the area of the upper triangles, noting that this will still overestimate the integral, since $x^n$ is concave-up on this interval. We thus have the inequality:

$$\frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^n - \frac{1}{2n} \sum_{r=1}^{n} \left[\left(\frac{r}{n}\right)^n - \left(\frac{r - 1}{n}\right)^n\right] > \int_0^1 x^n dx.$$

Simplifying, we see that

$$\frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^n - \frac{1}{2n} > \frac{1}{n + 1}$$

or simply

$$\sum_{r=1}^{n} \left(\frac{r}{n}\right)^n > \frac{3n + 1}{2n + 2}.$$
8. (Putnam 2003-A3) Find the minimum value of

\[ |\sin x + \cos x + \tan x + \cot x + \sec x + \csc x| \]

for real numbers \( x \). (Richard)

9. (Putnam 2004-A2) For \( i = 1, 2 \) let \( T_i \) be a triangle with side lengths \( a_i, b_i, c_i \), and area \( A_i \). Suppose that \( a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2 \), and that \( T_2 \) is an acute triangle. Does it follow that \( A_1 \leq A_2 \)? (Hint: Use Heron’s formula, and other solution can be found online.) (Lei)

10. (Putnam 2004-B2) Let \( m \) and \( n \) be positive integers. Show that

\[
\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.
\]

(Brett)