Solutions for Problem Set 6

1. (VT 1983) Let $f(x) = 1/x$ and $g(x) = 1 - x$ for $x \in (0, 1)$. List all distinct functions that can be written in the form $f \circ g \circ f \circ g \circ \cdots \circ f \circ g \circ f$ where $\circ$ represents composition. Write each function in the form $(ax+b)/(cx+d)$, and prove that your list is exhaustive. (`Tina`)

   Solution: By Linear Fractional Transformation, if $F(x) = \frac{ax+b}{cx+d}$ and $G(x) = \frac{ex+f}{gx+h}$ then $F \circ G(x) = \frac{Ax+B}{Cx+D}$ where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the multiplication of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

   So here we have $f(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $g(x) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $F_0 = f(x) = \frac{1}{x}$, $F_1 = (f \circ g)F_0$, $F_n = (f \circ g)F_{n-1}$. By LFT,

$$\begin{aligned} (f \circ g) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

   So using LFT, we find that

$$F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tfrac{1}{x}$$
$$F_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \tfrac{x}{x-1}$$
$$F_2 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = (x-1)$$
$$F_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \tfrac{1}{x}$$
$$F_4 = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} = \tfrac{x}{x-1}$$
$$F_5 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = (x-1)$$
$$F_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tfrac{1}{x}$$

   Since $F_6 = F_0$ and $F_n = (f \circ g)F_{n-1}$, the series is cyclical; and therefore the only distinct functions that $f \circ g \circ f \circ g \circ \ldots \circ f \circ g \circ f$ can take are $\{\frac{1}{x}, \frac{x}{x-1}, (x-1)\}$.

2. (a) (UIUC 2003 Mock) Let $f(x) = \dfrac{1}{1-x}$. Let $f_1(x) = f(x)$ and for each $n = 2, 3, \cdots$, let $f_n(x) = f(f_{n-1}(x))$. What is the value of $f_{2003}(2003)$?
   (b) (UIUC 1997) Let $x_1 = x_2 = 1$, and $x_{n+1} = 1996x_n + 1997x_{n-1}$ for $n \geq 2$. Find (with proof) the reminder of x_{1997} upon division by 3. (`Nicholas`) (Hint: (a) find the pattern; (b) find the periodic pattern modulo 3)

3. (UIUC 1997) Let $x_0 = 0$, $x_1 = 1$, and $x_{n+1} = \dfrac{x_n + nx_{n-1}}{n+1}$ for $n \geq 1$. Show that the sequence $\{x_n\}$ converges and finds its limit. (`Beth`)

   We begin by setting $b_n = \frac{-n}{n+1} * b_{n-1}$, for $n \geq 1$.

   Iterating this, we get:

$$b_n = (-1)^n \frac{n}{n+1} \frac{n-1}{n} *** \frac{2}{3} \frac{1}{2} d_0.$$

Hence,

$$x_n = x_0 + \sum_{k=0}^{n-1} d_k = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1}$$

.

Since we are given that $x_0 = 0$ and $x_1 = 1$, we get that $d_0 = x_1 - x_0 = 1$.

This leads to an alternating series with decreasing terms,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$$

and is convergent to $ln(1+1) = ln2$ because of the Taylor expansion for $ln(1+x)$.

Therefore the sequence $\{x_n\}$ converges with the limit ln2.

4. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 \neq 0$ and $a_{n+3} = 2a_{n+2} + 5a_{n+1} - 6a_n$. Find all possible values for $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. (`David Edmonson`)

   Solution. This is a third order linear recurrence relation. We are given that $\mathrm{a}_{n+3} = 2\mathrm{a}_{n+2} + 5\mathrm{a}_{n+1}$ - $6\mathrm{a}_n$. Thus, $\mathrm{a}_{n+3} = \mathrm{Aa}_n + \mathrm{Ba}_{n+1} + \mathrm{Ca}_{n+2}$, where A = -6, B = 5, and C = 2. Thus, the characteristic equation is $\lambda^3 = 2\lambda^2 + 5\lambda$ - 6. Solving λ^3 - $2\lambda^2$ - $5\lambda + 6 = 0$ we arrive at solutions: $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 =$ -2. Finding a general solution, we know that $\mathrm{a}_n = \mathrm{c}_1\lambda_1{}^n + \mathrm{c}_2\lambda_2{}^n + \mathrm{c}_3\lambda_3{}^n$. Thus, $\mathrm{a}_n = \mathrm{c}_1 1^n + \mathrm{c}_2 3^n + \mathrm{c}_3(\text{-}2)^n$. Since we are given no initial conditions, we are unable to determine the values of c_1, c_2, and c_3. However, we have not been asked to solve the general solution; we have been asked to find all possible values for $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Observe that $\frac{a_{n+1}}{a_n} = \frac{c_1 1^{n+1} + c_2 3^{n+1} + c_3(-2)^{n+1}}{c_1 1^n + c_2 3^n + c_3(-2)^n}$. Taking the limit as n $\to \infty$, we determine that the limit will be the λ with the largest magnitude, which in this case is $\lambda_2 = 3$. However, if the coefficient paired with this value (in this case, c_2) is equal to zero, then the limit will be the λ with the next largest magnitude, which is $\lambda_3 =$ -2, unless its coefficient is also equal to zero, in which case the limit will be last λ, which would be $\lambda_1 = 1$. Thus, all possible values for $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ are 3, -2, and 1.

5. The sequence x_n is defined by

$$x_1 = 2, \quad x_{n+1} = \frac{2 + x_n}{1 - 2x_n}.$$

   Prove that (a) $x_n \neq 0$ for all n; (b) x_n is not periodic. (`Richard`) (Hint: use matrix)

6. $a_1 = 0$, $a_{n+1} = 5a_n + \sqrt{24a_n^2 + 1}$. Prove that a_n is an integer for all n. (`David Rose`)

Solution: Computing the first few terms of the sequence we see that:

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 1 \\ a_3 &= 10 \\ a_4 &= 99 \\ a_5 &= 980. \end{aligned}$$

We then guess that the recursion satisfies the equation

$$a_n = 10a_{n-1} - a_{n-2}.$$

Note now that the sequence is defined by

$$a_{n+1}^2 - 10a_{n+1}a_n + a_n^2 - 1 = 0.$$

We also have

$$a_n^2 - 10a_n a_{n-1} + a_{n-1}^2 - 1 = 0.$$

We thus see that a_{n+1} and a_{n-1} are both roots to the polynomial equation

$$x^2 - 10a_n x + (a_n^2 - 1) = 0.$$

We thus have that

$$(x - a_{n+1})(x - a_{n-1}) = x^2 - 10a_n x + (a_n^2 - 1).$$

Expanding we see that $a_{n+1} + a_{n-1} = 10a_n$, or simply that $a_{n+1} = 10a_n - a_{n-1}$, as we wished to show. Since the integers are closed under addition and multiplication, we see that a_n is an integer for all n.

7. $a_0 = 1$, $a_1 = 5$, $a_n = \dfrac{2a_{n-1}^2 - 3a_{n-1} - 9}{2a_{n-2}}$. Prove that a_n is an integer for all n. (`Ben`)

8. (Putnam 1980-B3) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (express the answer in the simplest form) (`Derek`) (Hint: solve the recurrence)

9. (Putnam 1956-B6) $a_1 = 2$, $a_{n+1} = a_n^2 - a_n + 1$. (a) Prove that any two terms in $\{a_n\}$ are relatively prime; (b) Prove that $\sum_{n=1}^{\infty} 1/a_n = 1$. (`Lei`) (Hint: $b_n = a_n - 1$)

10. (Putnam 1993-A2) Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$. (`Erin`)

    Solution: Taking the hint, when we divide by z^4 we get, $z^8 + 4z^6 - 10z^4 + 4z^2 + 1 = z^4(z^4 + 4z^2 - 10 + \frac{4}{z^2} + \frac{1}{z^4}) = 0.$// The factored version implies that either 0 is a root of the original equation (obviously not true) or $z^4 + 4z^2 - 10 + \frac{4}{z^2} + \frac{1}{z^4} = 0$. Since this looks roughly like a binomial expansion, we use one of the formulas on the handout and check to see what $(z + \frac{1}{z})^4$ looks like. $(z + \frac{1}{z})^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$ which is remarkably close to what we are looking for. In fact, // $z^4 + 4z^2 - 10 + \frac{4}{z^2} + \frac{1}{z^4} = (z + \frac{1}{z})^4 - 16 = 0$. Thus, this problem simplifies to finding out when $(z + \frac{1}{z})^4 = 16$. This, in turn, simplifies to showing when $(z + \frac{1}{z}) = \pm 2$ and when $(z + \frac{1}{z}) = \pm 2i$. A lot of messy algebra eventually yields the roots ± 1 (each with a multiplicty of two), $i \pm i\sqrt{2}$, and $-i \pm i\sqrt{2}$ for the other four roots.