Problem Set 4

Discussion Problems

1. (UIUC 2003) Let $N = 9 + 99 + 999 + \cdots + \overbrace{99\cdots9}^{99}$. Determine the sum of digits of N. `(Derek)`

   Solution. First, notice that N is equivalent to

   $$10 + 100 + 1000 + ... + \overbrace{1000...00}^{100} \; - \; 99$$

   This equals

   $$\overbrace{1111...111110}^{100} \; - \; 99$$

   Which equals

   $$\overbrace{1111...111011}^{100}$$

   So the sum of the digits of N is 99.

2. (UIUC 1998) Evaluate $\displaystyle\sum_{k=1}^{n} \frac{k}{2^{k-1}}$. `(David Edmonson)`

   Solution. Here we are given the sum $1 + x + x^2 + ... + x^{n-1} = \dfrac{1 - x^n}{1 - x}$, or $\displaystyle\sum_{k=1}^{n} x^{k-1} = \frac{1 - x^n}{1 - x}$. Multiplying both sides by x gives us $\displaystyle\sum_{k=1}^{n} x^k = \frac{x(1 - x^n)}{1 - x} = \frac{x - x^{n+1}}{1 - x}$. Differentiating both sides gives us $\displaystyle\sum_{k=1}^{n} kx^{k-1} = \frac{(1 - x)(1 - (n + 1)x^n) + (x - x^{n+1})}{(1 - x)^2}$. However, we want $\displaystyle\sum_{k=1}^{n} \frac{k}{2^{k-1}} = \sum_{k=1}^{n} k(\frac{1}{2})^{k-1} = \sum_{k=1}^{n} kx^{k-1}$, where $x = \dfrac{1}{2}$. Substituting $x = \frac{1}{2}$ into $\dfrac{(1 - x)(1 - (n + 1)x^n) + (x - x^{n+1})}{(1 - x)^2}$ gives us a solution of $4 - 2(n + 1)(\dfrac{1}{2})^n - 4(\dfrac{1}{2})^{n+1}$.

3. (UIUC 2004) Let F_n denote the Fibonacci sequence, defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{F_n}{3^n}$. `(Nicholas)`

   Solution. Let S denote the sum:

$$S = \sum_{k=1}^{\infty} \frac{F_k}{3^k}$$

We can substitute $F_{k-1} + F_{k-2}$ for F_k.

$$= \sum_{k=1}^{\infty} \frac{F_{k-1} + F_{k-2}}{3^k}$$

We also know that $F_1 = 1$ and $F_2 = 1$, which allows us to compute the first two terms.

$$= \frac{1}{3} + \frac{1}{3^2} + \sum_{k=3}^{\infty} \frac{F_{k-1} + F_{k-2}}{3^k} = \frac{1}{3} + \frac{1}{9} + \sum_{k=3}^{\infty} \frac{F_{k-1}}{3^k} + \sum_{k=3}^{\infty} \frac{F_{k-2}}{3^k}$$

Rearranging indices and simplifying gives us

$$= \frac{1}{3} + \frac{1}{9} + \sum_{k=2}^{\infty} \frac{F_k}{3^{k+1}} + \sum_{k=1}^{\infty} \frac{F_k}{3^{k+2}} = \frac{1}{3} + \frac{1}{9} + \frac{1}{3} \sum_{k=2}^{\infty} \frac{F_k}{3^k} + \frac{1}{9} \sum_{k=2}^{\infty} \frac{F_k}{3^k}$$

$$= \frac{1}{3} + \frac{1}{9} + \frac{1}{3} \left(\sum_{k=1}^{\infty} \frac{F_k}{3^k} - \frac{1}{3} \right) + \frac{1}{9} \sum_{k=1}^{\infty} \frac{F_k}{3^k}$$

Recalling our sum S, we can substitute as follows and solve.

$$= \frac{1}{3} + \frac{1}{9} + \frac{1}{3} \left(S - \frac{1}{3} \right) + \frac{1}{9} S$$

Thus, $S = 3/5$

4. (VT 2003) Evaluate $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \cdots$. `(Ben)`

Solution: First, notice that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

But we need the negative of this, that is

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \tag{1}$$

Now we need to integrate equation (1).

$$-\int \ln(1-x)\, dx = \sum_{n=1}^{\infty} \int \frac{1}{n} x^n \, dx = \int \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \right) dx$$

The expression $-ln(1-x)$ can be integrated using the substitution

$$y = 1 - x,\ dy = -dx$$

to give

$$-\int \ln(1-x)\,dx = (1-x)\ln(1-x) - 1 + x$$

And integrating the rest of equation (1) using standard integrating techniques results in

$$(2) \qquad (1-x)\ln(1-x) - 1 + x = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots$$

Now, the left-hand side of equation (2) is almost what we need, except there is the additional factor of x in each expression. We can get rid of this factor by dividing equation (2) by x, to get

$$\frac{(1-x)\ln(1-x) - 1 + x}{x} = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots$$

as wanted.

5. (Putnam 1977 B-1) Evaluate the infinite product: $\prod_{n=1}^{\infty} \frac{n^3-1}{n^3+1}$. `(David Rose)`

Solution First note that $n^3 - 1 = (n-1)(n^2+n+1)$ and $n^3+1 = (n+1)(n^2-n+1)$. Our product then becomes

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \left(\prod_{n=2}^{\infty} \frac{n-1}{n+1}\right)\left(\prod_{n=2}^{\infty} \frac{n^2+n+1}{n^2-n+1}\right)$$

Now considering partial products we have

$$\prod_{n=2}^{k} \frac{n-1}{n+1} = \left(\frac{1}{3}\right)\left(\frac{2}{4}\right)\cdots\left(\frac{k-1}{k+1}\right) = \frac{2}{k(k+1)}$$

and

$$\prod_{n=2}^{k} \frac{n^2+n+1}{n^2-n+1} = \prod_{n=2}^{k} \frac{(n+1)^2-(n+1)+1}{n^2-n+1} = \frac{k^2+k+1}{3}$$

Thus

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \lim_{k\to\infty} \prod_{n=2}^{k} \frac{n^3-1}{n^3+1} = \lim_{k\to\infty} \left(\frac{2}{k(k+1)}\right)\left(\frac{k^2+k+1}{3}\right) = \frac{2}{3}.$$

6. (Putnam 1978 B-2) Express $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^2n + mn^2 + 2mn}$ as a rational number. `(Shelley)`

Solution: This fraction simplifies into the sum $\sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} \frac{1}{n(m+n+2)}$. Concentrating on the first sum we use partial fractions to obtain:

$$\frac{1}{n(m+n+2)} = \frac{1}{m+2}\left(\frac{1}{n} - \frac{1}{n+m+2}\right).$$

From here, if we substitute 1 for m, then we notice that it is a telescoping series and all but the terms $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}$ cancel out. For $m = 2$, we simply add $\frac{1}{4}$ to the group, and so on. As $m \to \infty$, we have $\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{m+2}\right) = S_{m+2}$.

Now we are left with $\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{1}{m+2}\right) S_{m+2}$. If we expand this out a bit, we see the following:

$m = 1 \quad \left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{2} + \frac{1}{3}\right)$

$m = 2 \quad \left(\frac{1}{2} - \frac{1}{4}\right)\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$

$m = 3 \quad \left(\frac{1}{3} - \frac{1}{5}\right)\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right)$

As $m \to \infty$, we obtain

$$1\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{4} + \frac{1}{5}\right) + \ldots + \frac{1}{m}\left(\frac{1}{m+1} + \frac{1}{m+2}\right)$$

We then rearrange this sequence into two telescoping series:

$$1 + \frac{1}{2} + \frac{1}{4} + \left(\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) \ldots + \left(\frac{1}{m} - \frac{1}{m+1}\right)\right)$$
$$+\frac{1}{2}\left(\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{m} - \frac{1}{m+2}\right)\right)$$
$$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \sum \frac{1}{m(m+1)} + \sum \frac{1}{m(m+2)}\right)$$
$$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + 1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{7}{4}.$$

7. (Putnam 1977 A-4) For $0 < x < 1$, express $\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$ as a rational function of x.

   (UIUC 2000) Evaluate $\frac{1}{2^1 - 2^{-1}} + \frac{1}{2^2 - 2^{-2}} + \frac{1}{2^4 - 2^{-4}} + \frac{1}{2^8 - 2^{-8}} + \cdots$ `(Tina)`

   Solution:

$$\begin{aligned} \sum_{n=0}^{N} \frac{x^{2^n}}{1 - x^{2^{n+1}}} &= \sum_{n=0}^{N}\left(\frac{1}{1 - x^{2^n}} - \frac{1}{1 - x^{2^{n+1}}}\right) \\ &= \frac{1}{1-x} - \frac{1}{1 - x^{2^{N+1}}} \to \frac{1}{1-x} - 1 = \frac{x}{1-x} \end{aligned}$$

as $N \to \infty$, since $|x| < 1$.

7b) Evaluate $\dfrac{1}{2^1 - 2^{-1}} + \dfrac{1}{2^2 - 2^{-2}} + \dfrac{1}{2^4 - 2^{-4}} + \dfrac{1}{2^8 - 2^{-8}} + \dots$

Solution:
Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{x^{-2^n} - x^{2^n}}$, so that the given series is $f(\frac{1}{2})$.

$$\begin{aligned} \frac{1}{x^{-2^n} - x^{2^n}} &= \frac{1}{x^{-2^n} - x^{2^n}} \frac{x^{2^n}}{x^{2^n}} \\ &= \frac{x^{2^n}}{1 - x^{2^{n+1}}} \\ &= \frac{x}{1-x} \qquad \text{from part (a)} \end{aligned}$$

Therefore, $f(\frac{1}{2}) = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$.

8. (Putnam 1984 A-2) Express $\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$ as a rational number. (`Erin`)

Solution: To begin, split the sum into partial fractions, you have, $\sum_{k=1}^{\infty} \frac{A}{3^k - 2^k} - \frac{B}{3^{k+1} - 2^{k+1}}$. Multiplying out the partial fractions and then factoring out common terms yields,
$3^k(3A - B) - 2^k(2A - B) = 6^k$
$3^k(3A - B) = 6^k + 2^k(2A - B)$
$3^k(3A - B) = (2^k)(3^k) + 2^k(2A - B)$
$3^k(3A - B) = (2^k)(3^k + 2A - B)$
$\frac{3^k}{2^k} = \frac{3^k + 2A - B}{3A - B}$

Solving for the numerator and the denominator gives $A = 2^k$ and $B = 2^{k+1}$. Thus, the sum has become $\sum_{k=1}^{\infty} \frac{2^k}{3^k - 2^k} - \frac{2^{k+1}}{3^{k+1} - 2^{k+1}}$. If you look at the partial sum (*i.e.* $\sum_{k=1}^{n} \frac{2^k}{3^k - 2^k} - \frac{2^{k+1}}{3^{k+1} - 2^{k+1}}$), it is clear that this is a telescoping series, thus you are left with,
$\sum_{k=1}^{n} \frac{2^k}{3^k - 2^k} - \frac{2^{k+1}}{3^{k+1} - 2^{k+1}} = 2 - \frac{2^{n+1}}{3^{n+1} - 2^{n+1}}$.

When you take $\lim_{n\to\infty} 2 - \frac{2^{n+1}}{3^{n+1} - 2^{n+1}}$, you get $2 - 0 = 2$ which is a rational number.

9. (Putnam 1997 A-3) Evaluate

$$\int_0^{\infty} \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) dx$$

(Richard)

Solution. Note that the series on the left is simply $x\exp(-x^2/2)$. By integration by parts,

$$\int_0^\infty x^{2n+1}e^{-x^2/2}dx = 2n\int_0^\infty x^{2n-1}e^{-x^2/2}dx$$

and so by induction,

$$\int_0^\infty x^{2n+1}e^{-x^2/2}dx = 2\times 4\times\cdots\times 2n.$$

Thus the desired integral is simply

$$\sum_{n=0}^\infty \frac{1}{2^n n!} = \sqrt{e}.$$

10. (Putnam 1999 A-4) Sum the series $\displaystyle\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{m^2n}{3^m(n3^m+m3^n)}$ (Frank)

Solution: Let $a_n = 3^n/n$, and $a_m = 3^m/m$. Then:

$$\begin{aligned}\frac{m^2n}{3^m(n3^m+m3^n)} &= \frac{m^2n}{ma_m(nma_m+mna_n)}\\ &= \frac{m^2n}{m^2na_m^2+m^2na_ma_n}\\ &= \frac{1}{a_m(a_m+a_n)}\end{aligned}$$

Note that by exchanging subscripts, we have:

$$\begin{aligned}S &= \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1}{a_m(a_m+a_n)}\\ &= \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{1}{a_n(a_m+a_n)}\end{aligned}$$

Thus:

$$\begin{aligned}2S &= \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1}{a_m(a_m+a_n)} + \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{1}{a_n(a_m+a_n)}\\ &= \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{a_m+a_n}{a_ma_n(a_m+a_n)}\\ &= \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1}{a_ma_n} = \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{nm}{3^{m+n}}\\ &= \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{n}{3^n}\frac{m}{3^m} = \left(\sum_{n=1}^\infty \frac{n}{3^n}\right)^2\end{aligned}$$

Recall the geometric series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, Substituting $x/3$ for x, we have $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \frac{3}{3-x}$.
Differentiating the series, we find that:

$$\sum_{n=0}^{\infty} \frac{nx^{n-1}}{3^n} = \frac{3}{(3-x)^2}. \tag{3}$$

Evaluating this formula at $x = 1$:

$$\sum_{n=0}^{\infty} \frac{n}{3^n} = \sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{(3-1)^2} = \frac{3}{4} \tag{4}$$

Thus, $2S = (3/4)^2$, so $S = 9/32$.