Problem Set 3

Discussion: Sept. 14, Sept. 16 (mostly on mathematical induction) The name after the problem is the designated writer of the solution of that problem. (Erin, Ben and Tina are exempted this week)

Discussion Problems

1. Prove that $\dfrac{n^5}{5}+\dfrac{n^4}{2}+\dfrac{n^3}{3}-\dfrac{n}{30}$ is an integer for $n=0,1,\cdots,$. `(Shelley)`

   Solution: Step 1: Set $n=1$.

$$\frac{1^5}{5}+\frac{1^4}{2}+\frac{1^3}{3}-\frac{1}{30}=1.\ \textbf{True}.$$

Step 2: Assume for $n=k$, $\dfrac{k^5}{5}+\dfrac{k^4}{2}+\dfrac{k^3}{3}-\dfrac{k}{30}$ is an integer, which we assume to be $x\in\mathbf{N}$, where $\mathbf{N}$ = set of all natural numbers. We then multiply by 30 to obtain

$$6k^5+15k^4+10k^3-k=30x$$

Step 3: Prove for

$$\frac{(k+1)^5}{5}+\frac{(k+1)^4}{2}+\frac{(k+1)^3}{3}-(k+1)\ \text{ is an integer.}$$

Multiplying by 30, we obtain $6(k+1)^5+15(k+1)^4+10(k+1)^3-(k+1) \vdots 30$. If we expand the polynomials (thanks to Pascal's handy triangle), then we obtain:

$$\begin{array}{rc} 6(k^5+5k^4+10k^3+10k^2+5k+1) & + \\ 15(k^4+4k^3+6k^2+4k+1) & + \\ 10(k^3+3k^2+3k+1) & - \\ (k+1) & \end{array}$$

Conveniently, if we regard each term with coefficients, they can factor out a greatest common denominator that when multiplied with the number outside the parenthesis equals 30.

$$\begin{array}{rcl} 30(k^4+2k^3+2k^2+k) & + & 6(k^5+1)+ \\ 30(2k^3+3k^2+2k) & + & 15(k^4+1)+ \\ 30(k^2+k) & + & 10(k^3+1)-(k+1) \end{array}$$

Because a term is divisible by a number if the parts of its sum is divisible by that number as well, then we no longer have to worry about these terms. However, we are left with

$$\begin{array}{rc} 6(k^5+1)+15(k^4+1)+10(k^3+1)-(k+1) & = \\ 6k^5+15k^4+10k^3-k+6+15+10-1 & = \\ \underline{6k^5+15k^4+10k^3-k}+30 & \end{array}$$

Notice that the underlined portion is equivalent to our original conjecture (from Step 2), which we assumed to be divisible by 30. And, of course, 30 is divisible by 30. Thus, all parts of our sum are divisible by 30. **Done**.

2. Show that, if n is odd, then $1^n + 2^n + \cdots + n^n$ is divisible by n^2. `(Lei)`

   Solution. First we'll rearrange the series to:

$$1^n + 2^n \ldots + n^n = (1^n + n^n) + (2^n + (n-1)^n) + \ldots + (m^n + (n-m)^n)$$

   Note that m exists because of the fact that n is odd. Then using binomial theorem we can expand these terms out, and we see that every term besides the first two terms will be multiples of n^2. So the first two terms are:

$$\binom{n}{1} 1^n * n^1 + \binom{n}{2} 2^{n-1} * n^2$$

   And we see they are obviously divisible by n^2.

3. Let r be a number such that $r + 1/r$ is an integer. Prove that for every positive integer n, $r^n + 1/r^n$ is an integer. `(Richard)`

   Solution: First note that $r^1 + (1/(r^1)) = 2$, which is an integer. Assume that $r^n + (1/(r^n))$ is an integer for all n. Now consider the product $(r + (1/r))$ $(r^n + (1/(r^n))$ $= (r^{n+1} + (1/r^{n+1})) + (r^{n-1} + (1/r^{n-1}))$ which is an integer by our induction hypothesis.

4. We need to put n cents of stamps on an envelop, but we have only (an unlimited supply of) 5 cents and 12 cents stamps. Prove that we can perform the task if $n \geq 44$. `(Beth)`

   Solution: We solve using induction. For the basis step, $n = 44$, we can use four 5 cent stamps and two 12 cent stamps, giving us $5 * 4 + 12 * 2 = 44$. We also need $n = 45$, $n = 46$, $n = 47$, and $n = 48$.

   For $n = 45$, we have 9 5-cent stamps

   For $n = 46$, we have 2 5-cent stamps and 3 12-cent stamps

   For $n = 47$, we have 7 5-cent stamps and 1 12-cent stamp

   For $n = 48$, we have 4 12-cent stamps.

   Notice that any integer greater than 48 can be expressed as $n = k + 5 * p$ where $k \in \{44, 45, 46, 47, 48\}$ and $p \geq 1$. Hence we can use the stamp combination for k given above and extra p 5-cent stamp to perform the required task.

5. Let n be a positive integer. Which one is larger: n^{n+1} or $(n+1)^n$? `(David Edmonson)`

   Solution. It will be shown that $\mathrm{n}^{n+1} > (\mathrm{n}+1)^n$ for all n$\geq$3. Let us prove this by induction. Consider the base case of n=3. Here we have: $3^{3+1} = 3^4 = 81 > (3+1)^3$ $= 4^3 = 64$. Thus the base case is true, and we have shown that $\mathrm{A}(3) = \frac{3^{3+1}}{(3+1)^3} > 1$. Now we proceed by induction. Suppose that for some k>3, $\mathrm{A(k)} = \frac{k^{k+1}}{(k+1)^k} > 1$. We want to prove: $\mathrm{A(k+1)} = \frac{(k+1)^{k+2}}{(k+2)^{k+1}} > 1$. Observe that we can write A(k+1) as A(k) $\times$ $\frac{A(k+1)}{A(k)}$. Write $\frac{A(k+1)}{A(k)}$ as $\frac{(k+1)^k (k+1)^{k+2}}{k^{k+1}(k+2)^{k+1}} = (\frac{(k+1)^2}{k(k+2)})^{k+1} = (\frac{k^2+2k+1}{k^2+2k})^{k+1}$. Since $\frac{k^2+2k+1}{k^2+2k}$

> 1, then $(\frac{k^2+2k+1}{k^2+2k})^{k+1} > 1$, or equivalently, $\frac{A(k+1)}{A(k)} > 1$. Since we are operating under the assumption that A(k) >1, then it follows that A(k+1) = A(k) $\times$ $\frac{A(k+1)}{A(k)} > 1$. This concludes the proof by induction, and hence we have proven that n^{n+1} > (n+1)^n for all n$\geq$3.

6. Show that for all n, $2^{3^n}+1$ is divisible by 3^{n+1}. `(Brett)`

   Solution: This statement is true when $n = 1$ since 9 divides 9.
   Assume the statement holds for $n = k$. Then,

$$\frac{2^{3^k}+1}{3^{k+1}} = m \text{ where } m \in \mathbf{N} \Rightarrow \quad 2^{3^k} = 3^{k+1}m - 1$$

   Then write,

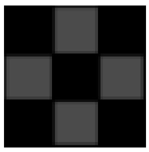
$$\frac{2^{3^{k+1}}+1}{3^{k+1+1}} = \frac{2^{3^k}2^{3^k}2^{3^k}+1}{3^{k+2}}$$

   Using the inductive hypothesis you get,

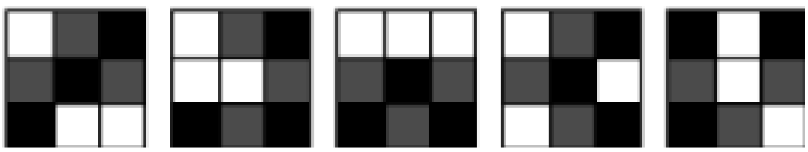
$$\frac{(3^{k+1}m-1)^3+1}{3^{k+2}} = \frac{(3^k3m)^3 - 3(3^k3m)^2 + 3(3^k3m)}{3 \cdot 3 \cdot 3^k}$$
$$= 3^{k+1}m^2(3^km-1) + m \in \mathbf{N}$$

7. Given a $(2m+1) \times (2n+1)$ checkerboard where the four corner squares are black, show that if one removes any one red and two black squares, the remaining board can be covered with dominoes (1×2 rectangles). `(Frank)`

   Solution: The proof proceeds by induction and exploits the many symmetries of the checkerboard. The smallest possible checkerboard (3×3) appears in the figure below:



   There are $\binom{5}{2}\binom{4}{1} = 40$ ways to remove two black squares and one red square from this board. However, only 5 of these are unique to a rotation or reflection. Those five appear in the figure below:



   Clearly each of these can be tiled with dominoes, so we have demonstrated S_1, our desired result for $m = n = 1$. Now suppose that S_k holds, i.e. a $(2m+1) \times (2n+1)$ checkerboard can be tiled with dominoes. We must show that this implies we can tile a checkerboard of dimensions $(2k+1) \times (2l+1)$, where $k \geq m$ and $l \geq n$. Since the checkerboard is symmetric under rotation, this is equivalent to showing that we can tile a checkerboard extended arbitrarily in any *one* direction.

We will introduce some notation to facilitate the remainder of the proof. Treat the board as a $(2m+1) \times (2n+1)$ matrix, and denote individual squares as $a(i,j)$ where $i = 1, 2, 3, \ldots, 2m+1$ and $j = 1, 2, 3, \ldots, 2n+1$ Denote $D[a(i,j), a(k,l)]$ as the domino covering those two respective squares.

Without loss of generality, consider incrementing n. We have a new checkerboard with dimensions $(2m+1) \times (2n+3)$, i.e. we have added two columns to the right side of the board. Now consider the locations of the three removed squares. As long as they fall in a sub-board of width $(2n+1)$ or smaller, we can tile the sub-board by the inductive hypothesis, and the remaining $(2m+1) \times 2$ rectangular region by the sequence

$$\begin{aligned}
D[a(1,j), a(1,j+1)] \\
D[a(2,j), a(2,j+1)] \\
D[a(3,j), a(3,j+1)] \\
\vdots \\
D[a(2m+1,j), a(2m+1,j+1)]
\end{aligned}$$

where j is either 1 or $2n+2$ depending on the locations of the squares we have removed.

There are only two cases in which we will be unable to use the inductive hypothesis directly as above:

- Exactly one square has been removed from the first two columns, and the remaining squares have been removed from the last two columns
- Exactly one square has been removed from the last two columns, and the remaining squares have been removed from the first two columns

Since the checkerboard is symmetric under reflection about the y-axis, we may consider without loss of generality the first case only. We will proceed by showing that the square removed from the first two columns can be transferred by tiling into the sub-board containing columns 3 through $2n+3$. This sub-board can be tiled using the inductive hypothesis.

We consider a board from which exactly one square has been removed from the first two columns, and exactly two squares have been removed from the last two columns.

Case I: Exactly one square was removed from the first column

In this case, we need not consider whether the square is black or white. Denote the removed square by $a(k,1)$. Tile the first two columns with the following sequence of dominos, recalling that there is a missing square at $a(k,1)$:

$$\begin{aligned} D[a(1,1),a(1,2)] \\ D[a(2,1),a(2,2)] \\ \vdots \\ D[a(k-1,1),a(k-1,2)] \\ D[a(k,2),a(k,3)] \\ D[a(k+1,1),a(k+1,2)] \\ \vdots \\ D[a(2m+1,1),a(2m+1,2)] \end{aligned}$$

This arrangement completely covers the first two columns of the board, and also covers exactly one square of the third column. This square is of the same color as the one removed from the first column, so we may treat it as a 'missing' square from the sub-board containing columns 3 through $2n+3$. This sub-board is hence tileable under the inductive hypothesis, so the entire board is tileable.

Case II: Exactly one square was removed from the second column
In this case, we must distinguish whether the removed square is black or white. In each intstance, we will tile the removed square into the sub-board containing columns 3 through $2n+3$ so that the 'missing' square from column 3 is of the same color as the one removed from the first column. By the same logic as above, this shows that the entire board is tileable. Denote the removed square by $a(k,2)$. When a black square has been removed, tile as follows:

$$\begin{aligned} D[a(1,1),a(1,2)] \\ D[a(2,1),a(2,2)] \\ \vdots \\ D[a(k-2,1),a(k-2,2)] \\ D[a(k-1,2),a(k-1,3)] \\ D[a(k-1,1),a(k,1)] \\ D[a(k+1,1),a(k+1,2)] \\ \vdots \\ D[a(2m+1,1),a(2m+1,2)] \end{aligned}$$

When a white square has been removed, if it is $a(1,2)$ tile as follows:

$$\begin{aligned} D[a(1,1), a(2,1)] \\ D[a(2,2), a(2,3)] \\ D[a(3,1), a(3,2)] \\ D[a(4,1), a(4,2)] \\ \vdots \\ D[a(2m+1,1), a(2m+1,2)] \end{aligned}$$

Otherwise, tile with the same sequence as when a black square has been removed.

Since the (3×3) checkerboard can be tiled by dominoes, and that fact that a $(2m+1) \times (2n+1)$ checkerboard can be tiled implies that a $(2m+1) \times (2n+3)$ checkerboard can be tiled, by the principle of mathematical induction, any checkerboard of dimensions $(2m+1) \times (2n+1)$ from which two black squares and one white square have been removed can be tiled with dominoes.

Q.E.D.

8. A group of n people play a round-robin tournament. Each game ends in either a win or a loss. Show that it is possible to label the players P_1, P_2, P_3, ..., P_n in such a way that P_1 defeated P_2, P_2 defeated P_3, ... , P_{n-1} defeated P_n. `(Nicholas)`

   Solution: It is clear that it is possible to rank the players when n=1, 2, 3. Now let us assume that it is possible to rank the players when n=k. We consider n=k+1. If we take k of the k+1 players, then by the inductive hypothesis we can rank them. This leaves us with one unranked player, call him x. If x loses to player k, then x takes on the ranking k+1, and we are done. If x defeats player k and defeats player 1, he takes on the ranking of 1, player 1 becomes number 2, and so on. If neither of the previous two cases occur, then we know that x has defeated at least one player, player k. So working backward, we look at each match-up. Once we find x's first loss, we know where to insert him. Assume x defeated player l+1, but lost to player l. Then we know to give x the ranking of l+1. Then l+1 becomes l+2, and so on till k, who becomes k+1.

9. (Putnam 2004 A3) Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

   for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.) `(David Rose)`

   Solution. We claim that $u_n = (n-1)(n-3)\cdots(2or1)$ for all $n \geq 3$, where the product ends in 2 if n is odd, and 1 if n is even. We will show this by induction. Note that the equation defining u_n is equivalent to

$$u_{n+3}u_n - u_{n+1}u_{n+2} = n!$$

We first check the case $n = 3$. By the recurrence relation we have, $u_3 u_0 - u_1 u_2 = 0!$ or equivalently $u_3 - 1 = 1$, so $u_3 = 2 = (3-1)$. We now assume that $u_n = (n-1)(n-3)\cdots(2or1)$. We then have

$$u_{n+1}u_{n-2} - u_n u_{n-1} = (n-2)!$$

which gives by the induction hypothesis

$$u_{n+1}(n-3)(n-5)\cdots(2 \text{ or } 1) - (n-1)! = (n-2)!$$

Rearranging and factoring then gives

$$u_{n+1} = n(n-2)(n-4)\cdots(2 \text{ or } 1),$$

completing our induction. Noting that $(n-1)(n-3)\cdots(2or1)$ is an integer for all n completes the proof.

10. If each person, in a group of n people, is a friend of at least half the people in the group, then it is possible to seat the n people in a circle so that everyone sits next to friends only. `(Derek)`

Solution.

Note: The definition of friendship in this problem should be made a bit clearer.

(a) Friendship is a commutative property (theoretically).
(b) A person is not friends with themselves, for the purposes of this problem.
(c) For each person, a_x, and out of the remaining people in the group, $N-1$, that person is friends with *MORE* than half of those people, where half is $(N-1)/2$, (The question's wording of "at least half" is not exactly correct.)

We shall define each person in the group by: $a_1,\ a_2,\ .\ .\ .\ a_{k-1},\ a_k$ where k is the integer index, given by N. We now look at the smallest case where a circle exists, N = 3, where N is the number of people in the group. Since each person has to sit next to both of the other two people, they must be friends with them for this to work. This is because if they were only friends with one of the other two people, they would be friends with only half of the remaining group, and we are told that they are friends with strictly more than that. So they must all be friends.

We can now construct a grouping system that represents the act of two people sitting next to each other; where, if two people are in the same parentheses, they are sitting next to each other. $(a_1, a_2), (a_2, a_3), (a_3, a_1)$

We now wish to prove that for any N, a circle can be constructed where only friends sit together given that each person is friends with more than half of the remaining people. Let us assume we have such a circle constructed for $N-1$ people. Now we ask, if we want to add one more person, where can they sit? Looking at the construction of the circle: $(a_1, a_2), (a_2, a_3),\ .\ .\ .\ (a_{k-1}, a_k), (a_k, a_1)$ Since there are exactly $(N-1)$ parentheses pairs, and the next person, call him/her a_x, must be friends with more than $(N-1)/2$ people, than there must be a parentheses pair (a_i, a_j) with which a_x is friends with both members. Now, simple make a new construction: $(a_1, a_2), (a_2, a_3),\ .\ .\ .\ (a_i, a_x), (a_x, a_j),\ .\ .\ .\ (a_{k-1}, a_k), (a_k, a_1)$ And you have successfully added a_x. Thus, $(N-1) \Rightarrow N$, and since we have $N = 3$ is true, By induction, it is proven that it is possible to seat any N people in a circle when each member is friends with more than half of the other members.