Problem set 2 solution

1. (Putnam, 1978-A1) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104. [Actually, 20 can be replaced by 19.] (Junping)

   **Solution.** Note that the numbers have the form $3n + 1$ for $n = 0, 1, \ldots, 33$. We seek $3n + 1, 3m + 1$ so that $n + m = 34$. Evidently $n = 0$ and $n = 17$ do not help. The other 32 numbers form 16 pairs with the required sum. Now we use Pigeon hole principle for the 16 pairs. So if we take 19 numbers then we are sure to get two from the same pair.

2. (Putnam, 2002-A2) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere. (Ben)

   **Solution:** We need to split the sphere in two, and this can best be done using a plane. To construct the plane, use the center of the sphere as a reference point, and any two other points that lie on the sphere. Since the hemisphere is closed, the two points that lie on the sphere belong to the hemisphere. Now, we have created two pigeonholes, namely the two hemispheres, and we still have three points to place on the sphere. Thus one hemisphere must contain two of those points, and thus four points must lie in one closed hemisphere.

   Since the hemisphere is closed, any of the remaining three points placed on the great circle defined by the intersection of the plane and the sphere are assumed to belong to whichever hemisphere is convenient.

3. 15 people sit around a table. When they sit down, they did not notice that a name tag is in front of each seat, and they found that no any name tag and the person sitting there match each other. Prove that after some rotation of the sitting order, at least two people will match the name tag where they sit. (Brett)

   **Solution.** Each of the fifteen people can be matched with their name simply by rotating everyone until that person is seated correctly. However, since no person starts in the correct seat, there are only fourteen rotations that can be used to match people to their names. Therefore, at least one rotation must match at least two people to their correct seat.

4. Given any $n + 1$ integers between 1 and $2n$, show that one of them is divisible by another. Is this best possible, *i.e.*, is the conclusion still true for $n$ integers between 1 and $2n$? (Brett)

   **Solution.** Let $a_1, a_2, \ldots, a_{n+1}$ be any integers between 1 and $2n$. Write $a_i = 2^h (2c_i + 1)$ for each $i$ between 1 and $n + 1$. Since the $a_i$’s are between 1 and $2n$, then $2c_i + 1 \in \{1, 3, \ldots, 2n - 1\}$. Hence, there exist $1 \leq i \neq j \leq n + 1$ such that $2c_i + 1 = 2c_j + 1$. Then $a_i$ divides $a_j$ or $a_j$ divides $a_i$. 

5. Six circles with radius 1 is randomly put inside of a circle with radius 6. Prove that at least one more circle with radius 1 can be put inside the big circle without intersecting the other six. (Beth)

**Solution.** The center of each of these circles (O1, O2, ..., O6) have to be within a circle of radius 5 inside the original circle P, call this new circle S. Since none of the circles are intersecting, then the minimum distance between centers is 2. Thus we can draw 6 circles of radius 2 (O1', O2', O3', O4', O5', O6'), all inside of S.

We then have to prove that (O1', ..., O6') does not cover the entire P. The area of the 6 circles are \( \pi \cdot 2^2 \), and there are 6 of them, so we have \( 24 \pi \). But the total area of S is \( \pi \cdot 5^2 = 25 \pi \). Thus there exists some point O7 in S, but not in the union of O1', ..., O6'.

![Figure 1: Problem 5](image)

6. A city has 10000 different telephone lines numbered by 4-digit numbers. More than half of the telephone lines are in the downtown. Prove that there are two telephone numbers in the downtown whose sum is again the number of a downtown telephone line. (Erin)

**Solution:** If 0000 is downtown then we are done. If not, let \( n_1 \) be the smallest telephone number downtown. Then define \( b_j \) such that \( b_j = n_i - n_1 \), where \( n_i \) is a downtown telephone number. Then there are at least 5000 possible \( b_j \) and 5001 \( n_i \) (since more than half of the telephone lines are downtown). This totals to 10001 which is more than 9999 thus there is at least 1 \( b_j \) and \( n_i \) such that \( b_j = n_i \).

7. Suppose a musical group has 11 weeks to prepare for opening night, and they intend to have at least one rehearsal each day. However, they decide not to schedule more than 12 rehearsals in any 7-day period, to keep from getting burned out. Prove that there exists a sequence of successive days during which the band has exactly 21 rehearsals. (Shelley)

**Solution:** We multiply the maximum number of games per week by the number of weeks until opening night to get the maximum number of games during this period:
11(12) = 132. We then use $a_i =$ total number of games in a given time period, so $1 \leq a_1 < ... < a_{77} \leq 132$. We then take some $b_i$ such that $b_i = a_i + 21$. This gives us the equation: $22 \leq b_1 < ... < b_{77} \leq 153$. We take the 77 numbers in the set $\{a_i\}$ and the 77 numbers in the set $\{b_i\}$ and combine then we have $|\{a_i\}| + |\{b_i\}| = 154$ elements. Yet we see that the total # of elements in the two sets are 1 through 153. So one element in $\{a_i\}$ must overlap with one of $\{b_i\}$, thus $a_i = b_j + 21$ for some $a_i, b_j$.

8. Prove that there exists a multiple of 2005 whose decimal expansion contains only digits 1 and 0. (Richard)

Solution: Let $A = 11111...111$ where there are 2005 1’s. Now consider all numbers of the form 1111...111 where the number of digits is less than 2005. By the pigeonhole principle, at least 2 (and many more in fact) of these numbers must have the same congruence class modulo 2005. Now take $A - B$, where $B$ is in the same congruence class as $A$, and $B < A$, and this is our multiple of 2005 as indicated.

9. (UIUC 2000) Suppose that $a_1, a_2, \cdots, a_n$ are $n$ given integers. Prove that there exist integers $r$ and $s$ with $0 \leq r < s \leq n$ such that $a_{r+1} + a_{r+2} + \cdots + a_s$ is divisible by $n$. (Richard)

Solution: Let $B_0 = 0$, and for $k = 1, 2, ..., n$ let $B_n = a_1 + a_2 + ... + a_n$. If any $B_k$ is divisible by $n$, then we are done, so assume that none does. By the pigeonhole principle, two of these $n+1$ integers $B_r$ and $B_s$ must leave the same remainder upon division by $n$. Hence $B_r - B_s = a_{r+1} + a_{r+2} + ... + a_s$ is a multiple of $n$ as required.

10. The Fibonacci sequence is defined by $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Prove that for any integer $m$, there exists $a_k$ such that $a_k$ ends with $m$ zeros. (David Edmonson)

Solution. This problem is essentially saying that $a_k \equiv 0 \pmod{10^m}$. Let us consider taking the last $m$ digits of each of the Fibonacci numbers and arranging them in pairs. For example, let us consider $m = 1$. Thus, we would have: (1,1),(1,2),(2,3),... Since there are a finite number of possible pairs, and this sequence of pairs goes on infinitely (since there are infinitely many Fibonacci numbers) then a pair must repeat itself (by the pigeon hole principle). However, remember that the Fibonacci numbers are defined recursively, such that each number is defined by the sum of the previous two numbers. This indicates that the sequence of pairs is cyclical. So once a repeated pair is found, the sequence can be traced eventually to a repetition of the pair $(1,1)$, which is also the first pair of the sequence. However, the way that the Fibonacci numbers are defined would mean that the number preceding the first 1 in this pair must be a 0, such that: $(...,0),(1,1)$, since $0 + 1 = 1$. Thus, it has been shown that for $m = 1$, there exists a Fibonacci number $a_k$ that ends with 1 zero. However, it follows from this logic that for any finite $m$ (and thus any positive integer) a Fibonacci number can be found that ends with $m$ zeros.

11. (a) A small party has six people. Each two people either know or don’t know each
other. Prove there are 3 people in the party such that either they all know each other, or nobody knows each other. (b) (VA Tech 2004-6) An enormous party has an infinite number of people. Each two people either know or don’t know each other. Given a positive integer $n$, prove there are $n$ people in the party such that either they all know each other, or nobody knows each other (so the first possibility means that if A and B are any two of the $n$ people, then A knows B, whereas the second possibility means that if A and B are any two of the $n$ people, then A does not know B). (Lei)

Solution. (a) Take a random person $A_1$. $A_1$ must at least know either three people or doesn’t know three people. WLOG we assume he does know three people and put these three people in a set named $S_1$. Of course everyone in this set knows $S_1$. Then we can choose $A_2$ from this set. If $A_2$ knows anyone else in $S_1$ then we’re done. If $A_2$ doesn’t know anyone then we have three people who don’t know each other since $A_2$ is arbitrary.

(b) Suppose first that there is an infinite subset $S$ such that each person only knows a finite number of people in $S$. Then pick a person $A_1$ in $S$. Then there is an infinite subset $S_1$ of $S$ containing $A_1$ such that $A_1$ knows nobody in $S_1$. Now choose a person $A_2$ other than $A_1$ in $S_1$. Then there is an infinite subset $S_2$ of $S_1$ containing $A_1,A_2$ such that nobody in $S_2$ knows $A_2$. Of course nobody in $S_2$ will know $A_1$ either. Now choose a person $A_3$ in $S_2$ other than $A_1$ and $A_2$. Then nobody from $A_1,A_2,A_3$ knows each other. Clearly we can continue this process indefinitely to obtain an arbitrarily large number of people who don’t know each other.

WLOG we can use the same process to prove that there are arbitrarily large number of people who all know each other.

12. (Larson page 29 1.5.7) (David Rose)

Solution. It is easy to verify the geometry in Figure ???. From there, considering the triangle with $L$ as its hypotenuse we have: $\sin \theta = \frac{x}{L}$. Considering the triangle with $x$ as its hypotenuse we have:

\[
\cos 2\theta = \frac{8 - x}{x} = \frac{8}{x} - 1
\]

Rearranging (??) gives:

\[
x = \frac{8}{\cos 2\theta + 1}
\]

and plugging (??) into $\sin \theta = \frac{x}{L}$ and rearranging gives the solution:

\[
L = \frac{8}{\sin \theta (\cos 2\theta + 1)}
\]
Figure 2: Problem 12