Problem set 2 solution

1. (Putnam, 1978-A1) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \cdots, 100$. Prove that there must be two distinct integers in A whose sum is 104. [Actually, 20 can be replaced by 19.] `(Junping)`

   Solution. Note that the numbers have the form $3n+1$ for $n = 0, 1, ..., 33$. We seek $3n+1$, $3m+1$ so that $n+m = 34$. Evidently $n = 0$ and $n = 17$ do not help. The other 32 numbers form 16 pairs with the required sum. Now we use Pigeon hole principle for the 16 pairs. So if we take 19 numbers then we are sure to get two from the same pair.

2. (Putnam, 2002-A2) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere. `(Ben)`

   Solution: We need to split the sphere in two, and this can best be done using a plane. To construct the plane, use the center of the sphere as a reference point, and any two other points that lie on the sphere. Since the hemisphere is closed, the two points that lie on the sphere belong to the hemisphere. Now, we have created two pigeonholes, namely the two hemispheres, and we still have three points to place on the sphere. Thus one hemisphere must contain two of those points, and thus four points must lie in one closed hemisphere.

   Since the hemisphere is closed, any of the remaining three points placed on the great circle defined by the intersection of the plane and the sphere are assumed to belong to whichever hemisphere is convenient.

3. 15 people sit around a table. When they sit down, they did not notice that a name tag is in front of each seat, and they found that no any name tag and the person sitting there match each other. Prove that after some rotation of the sitting order, at least two people will match the name tag where they sit. `(Brett)`

   Solution. Each of the fifteen people can be matched with their name simply by rotating everyone until that person is seated correctly. However, since no person starts in the correct seat, there are only fourteen rotations that can be used to match people to their names. Therefore, at least one rotation must match at least two people to their correct seat.

4. Given any $n+1$ integers between 1 and $2n$, show that one of them is divisible by another. Is this best possible, *i.e.*, is the conclusion still true for n integers between 1 and $2n$? `(Brett)`

   Solution. Let $a_1, a_2, \ldots, a_{n+1}$ be any integers between 1 and $2n$. Write $a_i = 2^{b_i}(2c_i + 1)$ for each i between 1 and $n+1$. Since the a_i's are between 1 and $2n$, then $2c_i + 1 \in \{1, 3, ..., 2n-1\}$. Hence, there exist $1 \le i \ne j \le n+1$ such that $2c_i + 1 = 2c_j + 1$. Then a_i divides a_j or a_j divides a_i.

5. Six circles with radius 1 is randomly put inside of a circle with radius 6. Prove that at least one more circle with radius 1 can be put inside the big circle without intersecting the other six. `(Beth)`

   Solution. The center of each of these circles (O1, 02,...,O6) have to be within a circle of radius 5 inside the original circle P, call this new circle S. Since none of the circles are intersecting, then the minimum distance between centers is 2. Thus we can draw 6 circles of radius 2 (O1', O2', O3', O4', O5', O6'), all inside of S.

   We then have to prove that (O1', ..., O6') does not cover the entire P. The area of the 6 circles are $\pi * 2^2$, and there are 6 of them, so we have $24 * \pi$. But the total area of S is $\pi * 5^2 = 25 * \pi$. Thus there exists some point O7 in S, but not in the union of O1',...,O6'.

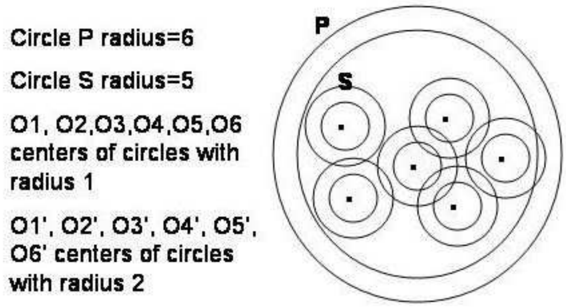

Figure 1: Problem 5

6. A city has 10000 different telephone lines numbered by 4-digit numbers. More than half of the telephone lines are in the downtown. Prove that there are two telephone numbers in the downtown whose sum is again the number of a downtown telephone line. `(Erin)`

   Solution: If 0000 is downtown then we are done. If not, let n_1 be the smallest telephone number downtown. Then define b_j such that $b_j = n_i - n_1$, where n_i is a downtown telephone number. Then there are at least 5000 possible b_j and 5001 n_i (since more than half of the telephone lines are downtown). This totals to 10001 which is more than 9999 thus there is at least 1 b_j and n_i such that $b_j = n_i$.

7. Suppose a musical group has 11 weeks to prepare for opening night, and they intend to have at least one rehearsal each day. However, they decide not to schedule more than 12 rehearsals in any 7-day period, to keep from getting burned out. Prove that there exists a sequence of successive days during which the band has exactly 21 rehearsals. `(Shelley)`

   Solution: We multiply the maximum number of games per week by the number of weeks until opening night to get the maximum number of games during this period:

$11(12) = 132$. We then use a_i = total number of games in a given time period, so $1 \leq a_1 < ... < a_{77} \leq 132$. We then take some b_i such that $b_i = a_i + 21$. This gives us the equation: $22 \leq b_1 < ... < b_{77} \leq 153$. We take the 77 numbers in the set $\{a_i\}$ and the 77 numbers in the set $\{b_i\}$ and combine then we have $|\{a_i\}| + |\{b_i\}| = 154$ elements. Yet we see that the total # of elements in the two sets are 1 through 153. So one element in $\{a_i\}$ must overlap with one of $\{b_i\}$, thus $a_i = b_j + 21$ for some a_i, b_j.

8. Prove that there exists a multiple of 2005 whose decimal expansion contains only digits 1 and 0. `(Richard)`

   Solution: Let $A = 11111....111$ where there are 2005 1's. Now consider all numbers of the form 1111...1111 where the number of digits is less than 2005. By the pigeonhole principle, at least 2 (and many more in fact) of these numbers must have the same congruence class modulo 2005. Now take $A - B$, where B is in the same congruence class as A and $B < A$, and this is our multiple of 2005 as indicated.

9. (UIUC 2000) Suppose that $a_1, a_2, \cdots, a_n$ are n given integers. Prove that there exist integers r and s with $0 \leq r < s \leq n$ such that $a_{r+1} + a_{r+2} + \cdots + a_s$ is divisible by n. `(Richard)`

   Solution: Let $B_0 = 0$, and for $k = 1, 2, ..., n$ let $B_n = a_1 + a_2 + ... + a_n$. If any B_k is divisible by n, then we are done, so assume that none does. By the pigeonhole principle, two of these $n+1$ integers B_r and B_s must leave the same remainder upon division by n. Hence $B_r - B_s = a_{r+1} + a_{r+2} + ... + a_s$ is a multiple of n as required.

10. The Fibonacci sequence is defined by $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Prove that for any integer m, there exists a_k such that a_k ends with m zeros. `(David Edmonson)`

    Solution. This problem is essentially saying that $a_k \equiv 0 \pmod{10^m}$. Let us consider taking the last m digits of each of the Fibonacci numbers and arranging them in pairs. For example, let us consider m = 1. Thus, we would have: (1,1),(1,2),(2,3),... Since there are a finite number of possible pairs, and this sequence of pairs goes on infinitely (since there are infinitely many Fibonacci numbers) then a pair must repeat itself (by the pigeon hole principle). However, remember that the Fibonacci numbers are defined recursively, such that each number is defined by the sum of the previous two numbers. This indicates that the sequence of pairs is cyclical. So once a repeated pair is found, the sequence can be traced eventually to a repetition of the pair (1,1), which is also the first pair of the sequence. However, the way that the Fibonacci numbers are defined would mean that the number preceding the first 1 in this pair must be a 0, such that: (...,0),(1,1), since 0 + 1 = 1. Thus, it has been shown that for m = 1, there exists a Fibonacci number a_k that ends with 1 zero. However, it follows from this logic that for any finite m (and thus any positive integer) a Fibonacci number can be found that ends with m zeros.

11. (a) A small party has six people. Each two people either know or don't know each

other. Prove there are 3 people in the party such that either they all know each other, or nobody knows each other. (b) (VA Tech 2004-6) An enormous party has an infinite number of people. Each two people either know or dont know each other. Given a positive integer n, prove there are n people in the party such that either they all know each other, or nobody knows each other (so the first possibility means that if A and B are any two of the n people, then A knows B, whereas the second possibility means that if A and B are any two of the n people, then A does not know B). `(Lei)`

Solution. (a) Take a random person A_1. A_1 must at least know either three people or doesn't know three people. WLOG we assume he does know three people and put these three people in a set named S_1. Of course everyone in this set knows S_1. Then we can choose A_2 from this set. If A_2 knows anyone else in S_1 then we're done. If A_2 doesn't know anyone then we have three people who don't know each other since A_2 is arbitrary.

(b)Suppose first that there is an infinite subset S such that each person only knows a finite number of people in S. Then pick a person A_1 in S. Then there is an infinite subset S_1 of S containing A_1 such that A_1 knows nobody in S_1. Now choose a person A_2 other than A_1 in S_1. Then there is an infinite subset S_2 of S_1 containing A_1,A_2 such that nobody in S_2 knows A_2. Of course nobody in S_2 will know A_1 either. Now choose a person A_3 in S_2 other than A_1 and A_2. Then nobody from A_1,A_2,A_3 knows each other. Clearly we can continue this process indefinitely to obtain an arbitrarily large number of people who dont know each other.

WLOG we can use the same process to prove that there are arbitrarily large number of people who all know each other.

12. (Larson page 29 1.5.7) `(David Rose)`

Solution. It is easy to verify the geometry in Figure **??**. From there, considering the triangle with L as its hypotenuse we have: $\sin\theta = \dfrac{x}{L}$. Considering the triangle with x as its hypotenuse we have:

$$\cos 2\theta = \frac{8-x}{x} = \frac{8}{x} - 1 \tag{1}$$

Rearranging (**??**) gives:

$$x = \frac{8}{\cos 2\theta + 1} \tag{2}$$

and plugging (**??**) into $\sin\theta = \dfrac{x}{L}$ and rearranging gives the solution:

$$L = \frac{8}{\sin\theta(\cos 2\theta + 1)} \tag{3}$$

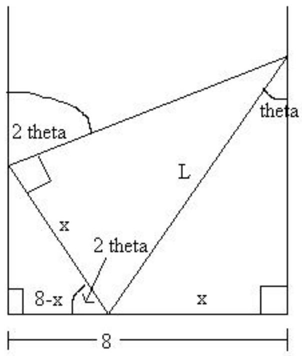

Figure 2: Problem 12