Problem Set 11 (last one!)

Discussion: Nov. 22, Nov. 29 (on linear algebra) The name after the problem is the designated writer of the solution of that problem. (Lei, Brett, and Richard are exempted this week)

Discussion Problems

1. (VT 1981) Let A be non-zero square matrix with the property that $A^3 = 0$, where 0 is the zero matrix, but with A being otherwise arbitrary. (a) Express $(I - A)^{-1}$ as a polynomial in A, where I is the identity matrix. (b) Find a 3×3 matrix satisfying $B^2 \neq 0$, $B^3 = 0$. (`Nicholas`)

2. (VT 1979) Let A be an $n \times n$ nonsingular matrix with complex elements, and let $\overline{A}$ be its complex conjugate. Let $B = A\overline{A} + I$, where I is the $n \times n$ identity matrix. (a) Prove or disprove: $A^{-1}BA = \overline{B}$. (b) Prove or disprove: the determinant of $A\overline{A} + I$ is real. (`Tina`)

   Solution:

   (a)

$$\begin{aligned} A^{-1}BA &= A^{-1}(A\overline{A} + I)A \\ &= A^{-1}A\overline{A}A + A^{-1}IA \\ &= \overline{A}A + I \end{aligned}$$

So $\overline{B}$ must equal $\overline{A}A + I$ for the original equation to hold. We need to show that $\overline{(AB)} = \overline{A}\overline{B}$ and that $\overline{(A+B)} = \overline{A} + \overline{B}$. However, this simplifies down to showing that the above equations hold simply for complex numbers A and B not whole matrices. This is true because matrix multiplication simply involves the addition and multiplication of complex numbers. So for complex numbers $A = a + bi$ and $B = c + di$, we have

$$\begin{aligned} \overline{A+B} &= \overline{(a+bi) + (c+di)} \\ &= \overline{(a+b) + (c+d)i} \\ &= (a+b) - (c+d)i \end{aligned}$$

$$\begin{aligned} \overline{AB} &= \overline{(a+bi)(c+di)} \\ &= \overline{ac + (ad+bc)i - bd} \\ &= (ac - bd) - (ad+bc)i \end{aligned}$$

$$\begin{aligned} \overline{A} + \overline{B} &= \overline{(a+bi)} + \overline{(c+di)} \\ &= (a - bi) + (c - di) \\ &= (a+b) - (c+d)i \end{aligned}$$

$$\begin{aligned} \overline{A}\,\overline{B} &= \overline{(a+bi)}\,\overline{(c+di)} \\ &= (a-bi)(c-di) \\ &= ac - (ad+bc)i - bd \\ &= (ac - bd) - (ad+bc)i \end{aligned}$$

So $\overline{B} = \overline{A}A + \overline{I} = \overline{A}A + I$.

(b) By a similar argument, the determinant of $A\overline{A} + I$ is real.

$$\begin{aligned} det(\overline{B}) &= det(A^{-1}BA) \\ &= det(A^{-1})det(B)det(A) \\ &= det(B) \end{aligned}$$

Since the determinant of a matrix is just found using multiplication and addition of its complex entries, $det(\overline{B}) = \overline{det(B)}$. This implies that $\overline{det(B)} = det(B)$ and this only way for this equality to hold is if $det(B)$ is real.

3. (VT 2003) Determine all invertible 2 by 2 matrices A with complex numbers as entries satisfying $A = A^{-1} = A'$, where A' denotes the transpose of A. (`Beth`)

4. (VT 2002) Let S be a set of 2×2 matrices with complex numbers as entries, and let T be the subset of S consisting of matrices whose eigenvalues are ± 1 (so the eigenvalues for each matrix in T are $\{1, 1\}$ or $\{1, -1\}$ or $\{-1, -1\}$). Suppose there are exactly three matrices in T. Prove that there are matrices A, B in S such that AB is not a matrix in S ($A = B$ is allowed). (`David Edmonson`)

   Solution. Let us label the three matrices in T as X, Y, and Z, and let I be the identity matrix. Let us suppose, by way of contradiction, that there are no A, B in S such that AB is not in S. If λ is an eigenvalue of X, then λ^r is an eigenvalue of X^r. Thus, the eigenvalues of X^2 are $\{1, 1\}$. So, the Jordan Canonical Form of X^2 is $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, where x = 0 or 1. If x = 1, then the matrices X^{2n} for n ≥ 1 are all different, but they are all members of T (because of their eigenvalues), but this is not possible because |T| = 3. So, x = 0, or in other words, $X^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, and thus $I \in T$. Thus we label the 3 elements of T as X, Y, and I, where I is the identity matrix and $X^2 = Y^2 = I$. Now consider XYX. We have $(XYX)(XYX) = XYX^2YX = XY^2X = X^2 = I$, and so the eigenvalues of XYX are ± 1, which means that $XYX \in T$, and so XYX must be either X, Y, or I. We now show this is impossible: If $XYX = X$, then $XYXXY = XXY = Y$, which would yield $X = Y$. If $XYX = I$, then $XXYXX = XX = I$, which would yield $Y = I$. However, neither of these options is possible because X, Y, and I are distinct. Finally, if $XYX = Y$, then $(XY)(XY) = I$, which means that $XY \in T$, which means that $XY = X$, Y, or I, and this can easily be shown to not be the case: If $XY = X$, then $XYX = XX = I$, but since we are working in the case where $XYX = Y$, then this would yield that $Y = I$, which is a contradiction. If $XY = I$, then $XYX = IX = X$, but since we are working in the case where $XYX = Y$, then this would yield that $Y = X$, which is a contradiction. Finally, if $XY = Y$, then $XYY = XY^2 = X$, but $XYY = YY = Y^2 = I$. This would yield that $X = I$, which is a contradiction. Thus, a contradiction is shown in all three cases, and hence the proof is completed. $\diamond$

5. (Putnam 1990-A5) If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$? (`Shelley`)

   Solution

   No, in cases $n \geq 3$ for an n x n matrix. We need only to find a counterexample.

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so $(AB)^2 = 0$

$BA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ so $(BA)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

There are other matrices that exist that may serve as counterexamples. A useful (and necessary) tool is to see that for the first matrix (A) with entries $i_1 j_1$ and the second matrix (B) with entries $i_2 j_2$, if you have some entry such that $j_1 = i_2$ for matrix A times matrix B, and that this holds true when (AB) is squared as well.

6. (Putnam 1969-B6) Let A and B be matrices of size 3×2 and 2×3 respectively. Suppose that the their product in the order AB is given by

$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}.$$

Show that the product BA is given by

$$BA = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

(Erin)

Solution: It is easy to show that AB has rank 2. Since the rank of BA must be at least as big as $A(BA)B = (AB)^2 = \begin{bmatrix} 72 & 18 & -18 \\ 18 & 45 & 36 \\ -18 & 36 & 45 \end{bmatrix} = 9AB$, which has rank 2, this implies that BA has rank 2 which means that BA is invertible. Then we have that:
$(BA)^3 = BABABA = B(AB)^2A = B(9AB)A = 9BABA = 9(BA)^2$
$(BA)^3 = 9(BA)^2$ and since BA is invertible, $(BA)^3(BA)^{-2} = 9(BA)^2(BA)^{-2}$
$BA = 9I$ which is what we were trying to show.

7. (Putnam 1994-A4) Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries. (Ben)

8. (Putnam 1996-B4) For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

(`David Rose`)

9. (Putnam 1992-B5) Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}.$$

Is the set $\{\frac{D_n}{n!}\}_{n \geq 2}$ bounded? (`Derek`)

10. (Putnam 1981-B4) A is a set of 5×7 real matrices closed under scalar multiplication and addition. It contains matrices of ranks $0, 1, 2, 4$ and 5. Does it necessarily contain a matrix of rank 3? (`Frank`)

Solution: No by counterexample. Consider the set of all matrices of the following form, where $a, b, c \in \mathbf{R}$:

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & b & c & 0 & 0 \end{bmatrix}$$

Clearly this set is closed under matrix addition and scalar multiplication. Setting $(a, b, c) = (1, 0, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0)$ yields matrices of rank 5, 4, 3, 2, and 1 respectively. Now suppose $a = 0$. Then the rank of the matrix is at most 2. When $a \neq 0$, the rank of the matrix is at least 4. Therefore, no matrix in this set has rank 3.