Problem Set 10

1. A and B start with $p = 1$. Then they alternately multiply p by one of the numbers 2 to 9. The winner is the one who first reaches (a) $p \geq 1000$, (b) $p \geq 10^6$. Who wins, A or B? (`Derek`)

2. (Putnam 1993-B2) Consider the following game played with a deck of 2n cards numbered from 1 to 2n. The deck is randomly shuffled and n cards are dealt to each of two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by 2n+1. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins? (`Brett`)

   Solution. Since the cards have values strictly less than $2n + 1$ there is at most one card that player A can play on any given turn that will allow him to win. However, since B has one more card than A on each of his turns, he has one card such that if he plays it, there is no card that A can play to win. Since B knows where all of the cards are he can determine which card this is and play it. Eventually the game will end with B playing the last card and winning the game. So the probability that A wins the game is 0.

3. (Putnam 1995-B5) A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking either (a) one bean from a heap, provided at least two beans are left behind in that heap, or (b) a complete heap of two or three beans. The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy. (`Nicholas`)

4. (Putnam 2002-B2) Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

   > Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

   Show that the player who signs first will always win by playing as well as possible. (`Beth`)

   Solution. We must first show that there must be a face with at least 4 edges. Begin by assuming that all faces have exactly 3 edges. So, if we let E = number of edges, and F = number of faces, and V = number of vertices, then we have the following equalities:

   $$2 * E = 3 * V$$

   since there are 3 edges at each vertex, which was given in the problem, and

   $$2 * E = 3 * F$$

   because it is a polyhedron and each edge has two faces.

   Euler's Formula is $F + V - E = 2$, and using this, and the two equalities we have above, we can solve for E=6, F=V=4. But we have and F > 4, so there must be a face with at least 4 edges.

So how can player 1 win? Player 1 will select a face (k) with at least 4 edges, and sign his name there. Then player 2 will pick a face to sign his name on. If player 2 does not pick a face adjacent to k, then there is nothing to worry about. If player 2 picked a face adjacent to the original face (k), then player 1 will pick the face that does not share any sides with the side player 2 just picked. The case where k has 4 edges is shown below.

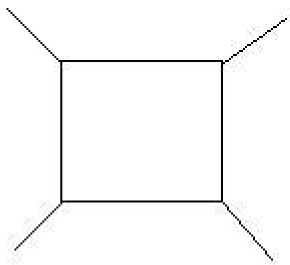

Figure 1: Diagram

Without loss of generality, if player 2 picked the top face, then player 1 would pick the bottom, so player two cannot block on both sides on his second turn. Then on player 1's third turn, he would win by signing the side that is left, and will have signed three sides that share a common vertex.

5. (Putnam 2002-B4) An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than 2/3. (`Lei`)

6. (SUMS 2003) Oscar and Nicole are playing a game with matchsticks: They form two piles of matches, one with 42 matches, and the other with 86. They take turns removing matches from the piles, according to the following rules: at each stage the matches taken (at least 1) must all come from one pile, and the number taken must be a divisor of the number of matches in the other pile. The player who removes the last match wins. Nicole goes first. Describe a strategy for Oscar to adopt so that he wins the game, no matter what Nicole does. Show that if we instead start with piles of 40 and 86 matches, then Nicole can always win, if she adopts the correct strategy. (`Frank`)

Solution: Let m be the number of matches in the first pile, and let n be the number of matches in the second pile. Likewise, let m' and n' be the respective numbers of matches after a turn has been taken. Define by $ord_2(k)$ the number of times that 2 divides k.

Lemma 1: If $ord_2(m) = ord_2(n)$ at the beginning of a player's turn, $ord_2(m') \neq ord_2(n')$ after this turn has been taken.

Proof: Without loss of generality suppose r matches are removed from m. Then r divides n, and $m' = m - r$ while $n' = n$. Let $v = ord_2(m) = ord_2(n)$, and write $m = 2^v m_1$, $n = 2^v n_1$, where m_1 and n_1 are the greatest odd factors of m and n respectively. Since r divides n, $ord_2(r) \leq v$.

Case I: Suppose $ord_2(r) = v$. Then we have $r = 2^v r_1$, where r_1 is the greatest odd factor of r. Thus $m' = 2^v m_1 - 2^v r_1 = 2^v(m_1 - r_1)$. Since m_1 and r_1 are are both odd, $(m_1 - r_1)$ is even, so 2 divides m' at least $v+1$ times. Thus $ord_2(m') \geq v+1 > ord_2(n')$, so $ord_2(m') \neq ord_2(n')$.

Case II: Suppose $u = ord_2(r) < v$. Then $r = 2^u r_1$ where r_1 is the greatest odd factor of r. Thus $m' = m - r = m - 2^u r_1 = 2^v m_1 - 2^u r_1 = 2^u(2^{v-u}m_1 - r_1)$. Since $2^{v-u}m_1$ is even and r_1 is odd, $2^{v-u}m_1 - r_1$ is odd, so $m' = 2^u(2^{v-u}m_1 - r_1)$ is even, and divisible by two exactly u times. Hence, $ord_2(m') = u < v = ord_2(n')$, so $ord_2(m') \neq ord_2(m')$.

Lemma 2: Suppose that at some point in the game $ord_2(m) \neq ord_2(m)$ at the end of one player's turn. Then the next player can choose her move such that $ord_2(m') = ord_2(n')$ at the end of her turn.

Proof: Suppose $m = 2^u m_1$, and $n = 2^v n_1$, where, without loss of generality, $u < v$, and m_1 and n_1 are the greatest odd factors of m and n, respectively. If she removes 2^u matches from pile n, then $m' = m$ and $n' = n - 2^u = 2^v n_1 - 1 = 2^u(2^{v-u}n_1 - 1)$. Since $2^{v-u}n_1 - 1$ is odd, $ord_2(n') = u = ord_2(m')$.

The game begins with $m = 42$, $n = 86$ on Nicole's turn. Thus $ord_2(m) = ord_2(n) = 1$. As a result, after Nicole's first turn $ord_2(m') \neq ord_2(n')$ by lemma 1. By lemma 2, Oscar can force m and n to statisfy $ord_2(m) = ord_2(n)$ at the beginning of Nicole's next turn. Continuing in this fashion, Oscar an ensure that $ord_2(m') \neq ord_2(n')$ at the end of each of Nicole's turns. Thus we cannot have $m = n = 0$ at the end of any of Nicole's turns. Therefore, Oscar can guarantee a win by the above strategy.

If we start with $m = 40$ and $n = 86$, Nicole should remove 2 matches from m. Then $ord_2(m) = ord_2(n)$ at the beginning of Oscar's first turn. If Nicole continues by playing according to the winning strategy given for Oscar above, she can guarantee a win by the same logic.

7. (Internet math puzzle) Sally and Sue have a strong desire to date Sam. They all live on the same street yet neither Sally or Sue know where Sam lives. The houses on this street are numbered 1 to 99.

   Sally asks Sam "Is your house number a perfect square?". He answers. Then Sally asks "Is is greater than 50?". He answers again.

   Sally thinks she now knows the address of Sam's house and decides to visit.

   When she gets there, she finds out she is wrong. This is not surprising, considering Sam answered only the second question truthfully.

Sue, unaware of Sally's conversation, asks Sam two questions. Sue asks "Is your house number a perfect cube?". He answers. She then asks "Is it greater than 25?". He answers again.

Sue thinks she knows where Sam lives and decides to pay him a visit. She too is mistaken as Sam once again answered only the second question truthfully.

If I tell you that Sam's number is less than Sue's or Sally's, and that the sum of their numbers is a perfect square multiplied by two, you should be able to figure out where all three of them live. (`Tina and Erin`)

Solution:

The following table shows Sam's possible address based on his answers to Sally's questions:

	Perfect Square	Not a Perfect Square
Less than or Equal to 50	$1, 4, 9, 16, 25, 36, 49$	$2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 50$
Greater than 50	$64, 81$	$51, 52, 53, 54, 55, 56, 57, 58, 59, 60$ $61, 62, 63, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99$

The only way that Sally could have any idea where Sam lives is if 1) he answered that his address was a perfect square greater than 50 and 2) Sally's address is also a perfect square greater than 50. So she went to either 64 or 81 and lives in the other. Since Sam only answered the second question truthfully, his address is greater than 50, but not a perfect sqaure. Similarly, the following table shows Sam's possible address based on his answers to Sue's questions:

	Perfect Cube	Not a Perfect Cube
Less than or Equal to 25	$1, 8$	$2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25$
Greater than 25	$27, 64$	$27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50$ $51, 52, 53, 54, 55, 56, 57, 58, 59, 60$ $61, 62, 63, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99$

The only way that Sue could have any idea where Sam lives is if he answered that his address was a perfect cube. And since we know that he answered the second question truthfully, his address is greater than 25 (since he have already determined it to be greater than 50). Therefore, Sue has two options: 27 and 64. The only way she would have a guess as to which one he lived in is if she lived in the other one.

We now know that Sam's address is greater than 50, but neither a perfect square nor a perfect cube. Since Sally and Sue's addresses are both greater than Sam's, and Sam's address is greater than 50, Sue must live in 64. This means that Sally must live in 81. Since the sum of all three address is a perfect square multiplied by 2, we can determine Sam's address. Their sum must lie on the interval $[196, 208]$ $((64 + 81 + 51), (64 + 81 + 63)$, since Sam's address must be greater than 50 but less than 64). The only perfect square multiplied by 2 on this interval is $200 = 2(10^2)$. Therefore, Sam's address is $200 - 64 - 81 = 55$.

8. (Internet math puzzle) Mr. S. and Mr. P. are both perfect logicians, being able to correctly deduce any truth from any set of axioms. Two integers (not necessarily unique) are somehow chosen such that each is within some specified range. Mr. S. is given the sum of these two integers; Mr. P. is given the product of these two integers. After receiving these numbers, the two logicians do not have any communication at all except the following dialogue:

   (a) Mr. P.: I do not know the two numbers.
   (b) Mr. S.: I knew that you didn't know the two numbers.
   (c) Mr. P.: Now I know the two numbers.
   (d) Mr. S.: Now I know the two numbers.

   Given that the above statements are absolutely truthful, what are the two numbers? (`Davis Rose and Richard`)