Problem set 1 solution

1. (VT, 1992-2) Assume that $x1 > y1 > 0$ and $y2 > x2 > 0$. Find a formula for the shortest length l of a planar path that goes from $(x1, y1)$ to $(x2, y2)$ and that touches both the x-axis and the y-axis. Justify your answer. `(Brett)`

   Solution: Use the physics principles that state that a particle will travel along a shortest path and that any angles of incidence along this path are equal to the angles of reflection. To construct such a path between points (x_1, y_1) and (x_2, y_2) reflect (x_1, y_1) about the x-axis and (x_2, y_2) about the y-axis. Then draw a straight line between the two points. So the length of the shortest path is given by $\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$.

2. (VT, 1984-2) Consider any three consecutive positive integers. Prove that the cube of the largest cannot be the sum of the cubes of the other two. `(Beth)`

   Solution: We give the proof by contradiction. Assume that there are 3 consecutive integers and that the cube of the largest is the sum of the cubes of the other two. We assume that the 3 consecutive integers are $x - 1$, x and $x + 1$. Then $(x + 1)^3 = x^3 + (x - 1)^3$. Expanding this out we get $x^3 + 3x^2 + 3x + 1 = x^3 + x^3 - 3x^2 + 3x - 1$, and it simplifies $2 = x^2(x - 6)$. Since $x - 6$ is an integer factor of 2, and $x^2 > 0$, now we have two cases: Case 1: $x - 6$ is 1, then $x = 7$, and thus not true because $8^3 \neq 6^3 + 7^3$; Case 2: $x - 6$ is 2, then $x = 8$, and thus not true because $9^3 \neq 7^3 + 8^3$. So we have a contradiction.

3. (VT, 1989-2) Let A be a 3×3 matrix in which each element is either 0 or 1 but is otherwise arbitrary. (a) Prove that $det(A)$ cannot be 3 or -3. (b) Find all possible values of $det(A)$ and prove your result. `(Erin)`

   Solution: Let A be a 3×3 matrix in which each element is either 0 or 1 but is otherwise arbitrary.

   (a) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

   $$det(A) = a_{11}det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{21}det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{31}det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}.$$

   The only possibilities for the 2×2 matrices with entries 0 or 1 are $\{-1, 0, 1\}$. Hence, if a_{11}, a_{21}, or $a_{31} = 0$ then $-2 \leq det(A) \leq 2$. The only way for $det(A) = 3$ or -3 is if $a_{11} = a_{21} = a_{31} = 1$. For $det(A) = 3$,
   $det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} = 1$ and $det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} = -1$,
   but the only possible matrix that makes the first two 1 does not yield a -1 for the last matrix. Thus $det(A) \neq 3$. The only possible way for $det(A) = -3$ is if
   $det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} = -1$ and $det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} = 1$,
   however, the only matrix that yields a -1 for the first two matrices does not have 1 as the last determinant. Thus, $det(A) \neq -3$.
   (b) $det(A)$ can assume any of the integer values between -2 and 2. $det(A) = 0$ when A is

the zero matrix; $det(A) = 1$ if $A = I_3$, the identity matrix, and $det(A) = -1, 2, -2$ if A is the following matrices respectively:

$$A_{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_{-2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

4. (Putnam, 1993-A1) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the Figure. Find c so that the areas of the two shaded regions are equal. `(Shelley)`

   Solution: The line $y = c$ bisects $y = x - 3x^3$ at points a and b. Assuming that the two "shaded" areas are equal, then the integral of the function $y = 2x - 3x^3$ minus c, in the interval $[0, b]$, would equal 0. If we take the integral $\int_0^b (2x - 3x^3 - c)dx$, then we end up with: $b^2 - (3/4)b^4 - bc = 0$. We can also set $c = 2b - 3b^3$. Substituting this in to our original integral, we obtain: $b^2 - (3/4)b^4 - 2b^2 + 3b^4 = 0$. Simplified, we find that $b = 2/3$, and therefore $c = 4/9$.

5. (Putnam, 2001-A1) Consider a set S and a binary operation $*$, *i.e.*, for each $a, b \in S$, $a*b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$. `(Frank)`

   Solution: By the definition of a binary operation, $a * b \in S$. By the above property of this particular binary operation we may thus write $((b * a) * b) * (b * a) = b$. However, by the same property, we have $(b * a) * b = a$. Substituting, $a * (b * a) = b$.

6. (Putnam, 2002-B1) Shanille OKeal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots? `(David Edmonson)`

   Solution. We will let $P(m \mid k)$ = probability that Shanille hits m free throws after k tries. We begin by considering k trials for small values of k. When k = 2, we have P(1|2) = 1 (given). For k = 3, we have P(1|3) = P(2|3) = $\frac{1}{2}$, since it is given that the probability that she makes the third shot is equal to the proportion of shots that she has hit so far, which is $\frac{1}{2}$. For k = 4, we must consider the following cases, where M refers to a miss and H refers to a hit: HMHH, HMHM, HMMH, and HMMM. Here, P(HMHH) = $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$; P(HMHM) = $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$; P(HMMH) = $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$; and P(HMMM) = $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$. Thus, P(1|4) = $\frac{1}{3}$, P(2|4) = $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$, and P(3|4) = $\frac{1}{3}$. Examining the case of k = 5 similarly yields P(1|5) = P(2|5) = P(3|5) = P(4|5) = $\frac{1}{4}$. We are led to guess that P(m|k) = $\frac{1}{k-1}$ for all m = 1, 2, ... , k-1. This will be proven by using induction on k. We have already shown this to be true for k = 2, 3, 4, and 5. Now suppose that it is true for k-1. For k, we would then have P(1|k) = $\dfrac{1}{k-2} \times \frac{k-2}{k-1}$ = $\frac{1}{k-1}$. For m>1, we have P(m|k) = $\frac{1}{k-2} \times \frac{k-1-m}{k-1} + \frac{m-1}{k-1} = \frac{1}{k-1}$. Hence, by induction, the result is shown to be true for k. So, the probability that Shanille hits exactly 50 of her first 100 shots is P(50|100) = $\frac{1}{100-1} = \frac{1}{99}$.

7. (UIUC, 2005-2) Evaluate the integral $I = \int_0^\pi \ln(\sin x)dx$. `(Lei)`

   Solution: From the symmetry $\sin(x) = \sin(\pi - x)$, we have $I = 2\int_0^{\pi/2} \ln(\sin x)dx$; from another symmetry $\cos(x) = \sin(\pi/2 - x)$, we can find that $\int_0^{\pi/2} \ln(\cos x)dx = \int_0^{\pi/2} \ln(\sin x)dx$.

Thus $I = 2\int_0^{\pi/2} \ln(\sin x)dx = \int_0^{\pi/2} \ln(\cos x)dx + \int_0^{\pi/2} \ln(\sin x)dx$. From the trigonometry formula $\sin(2x) = 2\sin x\cos x$, we have $\int_0^{\pi/2}[\ln(\cos x)dx + \ln(\sin x)]dx = \int_0^{\pi/2}[\ln(2\sin x\cos x) - \ln 2]dx = \int_0^{\pi/2}[\ln(\sin(2x)) - \ln 2]dx$. But $\int_0^{\pi/2} \ln(\sin(2x))dx = (1/2)\int_0^{\pi} \ln(\sin y)dy = (1/2)I$ from change of variable $y = 2x$. Thus $I = I/2 - \int_0^{\pi/2} \ln 2dx = I/2 - \ln 2\pi/2$, and $I = -\pi\ln 2$.

8. (MIT training 2004-12, one star) Let p and q be consecutive odd primes (i.e., no prime numbers are between them). Show that $p + q$ is a product of at least three primes. For instance, $23 + 29$ is the product of the three primes 2, 2, and 13. `(Nicholas)`

   Solution: Consider to consecutive odd primes p and q. Since p and q are odd, it follows that their sum (p + q) is even. Hence, p + q is the product of 2 and (p + q)/2. We know that 2 is prime. Thus, all that remains is to show that (p + q)/2 is not prime. Here we use the fact that p and q are consecutive odd primes. Since p and q are consecutive primes, it follows that (p + q)/2, which is the arithmetic mean of p and q, is not prime.

9. (Berkeley training) Given an infinite number of points in a plane, prove that if all the distances between them are integers, then the points are all on a straight line. `(Derek)`

   Solution. Suppose the given set contains three non-collinear points, say A, B and C, such that $|AB| =: r$ and $|AC| =: s$ are integers. If P is any point at integral distance from both A and B, then by the triangle inequality $|PA - PB|$ is one of the integers $0,\ 1,\ \ldots,\ r$. Hence P must lie on one of the hyperbolas

   $$H_j := \{X\ :\ |XA\ -\ XB|\ =\ j\} \qquad j\ =\ 1,\ 2,\ \ldots,\ r-1,$$

   on the line H_r through the points A and B outside the segment AB itself, or on the perpendicular bisector H_0 of AB. Analogously, a point P an integral distance from both A and C must be on one of the hyperbolas

   $$K_i := \{X\ :\ |XA\ -\ XC|\ =\ i\} \qquad i\ =\ 1,\ 2,\ \ldots,\ s-1,$$

   on the line K_s through A and C or on the perpendicular bisector K_0 of AC. Any point in our set must be in one of the sets $H_i \bigcap K_i$. Since the lines AB and AC are different, none of the sets Hi coincides with a set K_j. Thus, for all j and i, $H_i \bigcap K_i$ is the intersection of two distinct curves of degree at most two, so it contains at most 4 points. It cannot be made of lines that coincide, which would give us an infinite number of points. Therefore the given set contains at most $4(\ r+\ 1)(s\ +\ 1)$ points. This gives us a contradiction, so all of the points must lie on the same line.

10. (Northwestern training) Believe it or not the following function is constant in an interval [a, b]. Find that interval and the constant value of the function. `(Tina)`

    $$f(x) = \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$$

Solution: Consider

$$
\begin{aligned}
x + 2\sqrt{x-1} &= (x-1) + 2\sqrt{x-1} + 1 \\
&= (\sqrt{x-1} + 1)^2
\end{aligned}
$$

In a similar manner, we can also complete the square for the second square root in $f(x)$. So

$$
\begin{aligned}
f(x) &= \sqrt{(\sqrt{x-1}+1)^2} + \sqrt{(\sqrt{x-1}-1)^2} \\
&= (\sqrt{x-1}+1) \pm (\sqrt{x-1}-1)
\end{aligned}
$$

<u>Case 1:</u>

$$
\begin{aligned}
f(x) &= (\sqrt{x-1}+1) + (\sqrt{x-1}-1) \\
&= 2\sqrt{x-1}
\end{aligned}
$$

But this function is not constant over an interval.
<u>Case 2:</u>

$$
\begin{aligned}
f(x) &= (\sqrt{x-1}+1) - (\sqrt{x-1}-1) \\
&= 2
\end{aligned}
$$

which is a constant function for

$$
\begin{array}{cccc}
 & \sqrt{x-1}-1 & \leq 0 \\
 & \sqrt{x-1} & \leq 1 \\
\text{-1}\leq & (x-1) & \leq 1 \\
0 \leq & x & \leq 2
\end{array}
$$