A review of stability and dynamical behaviors of differential equations:

scalar ODE: $u_t = f(u)$, system of ODEs: $\begin{cases} u_t = f(u,v), \\ v_t = g(u,v), \end{cases}$

reaction-diffusion equation:
$u_t = D\Delta u + f(u)$, $x \in \Omega$, with boundary condition

reaction-diffusion system:
$\begin{cases} u_t = D_u \Delta u + f(u,v), \\ v_t = D_v \Delta v + g(u,v), \end{cases}$, $x \in \Omega$, with boundary condition

All equation is in form of $U_t = F(U)$, where u can be a scalar or vector, spatial independent or dependent

Abstract Equation $U_t = F(U)$

Equilibrium solution: U_0 such that $F(U_0) = 0$,
linearized operator: $F'(U_0)$

U_0 is **stable** if the eigenvalues of equation $F'(U_0)w = \lambda w$ are all with negative real parts.

(Linear behavior) Since the equation $U_t = F(U)$ is approximately the linearized equation $U_t = F'(U)(U - U_0)$ near the equilibrium solution $U = U_0$, then $U(t) \approx \sum C_i \exp(\lambda_i t)\phi_i$ near $U = U_0$, where (λ_i, ϕ_i) are the eigenvalue-eigenvector pairs of $F'(U_0)w = \lambda w$.

If $\lambda_1 = \max \lambda_i$, then $U(t) \approx C_1 \exp(\lambda_1 t)\phi_1$ if $U(0) \approx U_0$.

Turing bifurcations in reaction-diffusion system

$u_t = u_{xx} + \lambda f(u,v),\ v_t = dv_{xx} + \lambda g(u,v),$
$u_x(t,0) = u_x(t,\pi) = 0,\ v_x(t,0) = v_x(t,\pi) = 0$

If (u_0, v_0) is an equilibrium so that $f(u_0,v_0) = g(u_0,v_0) = 0$, then (u_0,v_0) is also an equilibrium of $u_t = f(u,v),\ v_t = g(u,v)$.

Linearized operator: ODE: Jacobian $J = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$

PDE: $diag(d_u\Delta, d_v\Delta) + J = \begin{pmatrix} \Delta & 0 \\ 0 & d\Delta v \end{pmatrix} + \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$

Eigenvalue problem: $\begin{cases} \Delta\phi + \lambda f_u(u_0,v_0)\phi + \lambda f_v(u_0,v_0)\psi = \mu\phi, \\ d\Delta\psi + \lambda g_u(u_0,v_0)\phi + \lambda g_v(u_0,v_0)\psi = \mu\psi, \\ \phi_x(0) = \phi_x(\pi) = 0, \ \ \psi_x(0) = \psi_x(\pi) = 0. \end{cases}$

If all eigenvalues of J are with negative real parts (so (u_0, v_0) is a stable equilibrium solution for ODE), is (u_0, v_0) also a stable equilibrium solution for PDE?

It seems so since the additional part is consist of diffusion operators only, and diffusion is supposed to stabilizing......

But, as Alan Turing pointed out, (u_0, v_0) could be an unstable equilibrium solution for PDE even if it is stable for ODE! So diffusion has an unstable effect for such system.

How is that possible? Let's calculate now......

Ideas: Eigenfunction could be in form of $\mathbf{V}\cos(mx)$, where $\cos(kx)$ is the eigenfunction of $\phi'' = \mu\phi$, $\phi'(0) = \phi'(\pi) = 0$, and **V** is a vector.

Substituting $(\phi, \psi) = \mathbf{V}\cos(kx)$ into the equation, we get

$$(\lambda J - k^2 D)\mathbf{V} = \mu\mathbf{V},$$

where $J = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$. Thus the eigenvalues of the reaction-diffusion system are $\mu_{1,k}$, $\mu_{2,k}$, which are eigenvalues of $J - k^2 D$, with $k = 1, 2, \cdots$.

Question: Can some of $\mu_{1,k}$, $\mu_{2,k}$ be positive? Which one is the largest?

Classification of linear system $Y' = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} Y$:

Two real eigenvalues:

1. $\lambda_1 > \lambda_2 > 0$: source
2. $\lambda_1 > \lambda_2 = 0$: degenerate source
3. $\lambda_1 > 0 > \lambda_2$: saddle
4. $\lambda_1 = 0 > \lambda_2$: degenerate sink
5. $0 > \lambda_1 > \lambda_2$: sink

Two complex eigenvalues: $\lambda_\pm = a \pm bi$

1. $a > 0$: spiral source
2. $a = 0$: center
3. $a < 0$: spiral sink

One real eigenvalue: $\lambda_1 = \lambda_2 = \lambda$
1. $\lambda > 0$: star source or “trying to spiral source”
2. $\lambda = 0$: parallel lines
3. $\lambda < 0$: star sink or “trying to spiral sink”

Generic Cases: (most likely, not fragile, and nonlinear system behaves similar to linear system)

Source, Sink, Saddle, Spiral source, Spiral sink

Stable: sink or spiral sink (real part of eigenvalues is negative)
Unstable: source, spiral source, saddle (at least one of eigen-value has positive real part)

Trace-determinant criterion for stability:

$$\lambda^2 - A\lambda + B = 0$$

$A^2 - 4B > 0$: two real eigenvalues
(1) $A > 0$, $B > 0$: source
(2) $A < 0$, $B > 0$: sink
(3) $B < 0$: saddle

$A^2 - 4B < 0$: two complex eigenvalues
(1) $A > 0$: spiral source
(2) $A < 0$: spiral sink

Stable: $A < 0$ and $B > 0$
Unstable: $A < 0$ and $B > 0$, or $B < 0$

Calculations:

$$\lambda\mu I - [J - k^2 D] = \begin{pmatrix} \mu - (\lambda f_u - k^2) & -\lambda f_v \\ -\lambda g_u & \mu - (\lambda g_v - k^2 d) \end{pmatrix}$$

Equation of eigenvalues:
$\mu^2 - [\lambda(f_u + g_v) - k^2(1+d)]\mu + [\lambda^2(f_u g_v - f_v g_u) - k^2(d f_u + g_v)\lambda + k^4 d] = 0$
or $\mu^2 - A\mu + B = 0$
where $A = \lambda f_u + \lambda g_v - k^2 - k^2 d$
$B = \lambda^2(f_u g_v - f_v g_u) - k^2(d f_u + g_v)\lambda + k^4 d$

We want it to be unstable, thus either $A < 0$ and $B > 0$, or $B < 0$, but we also want it is stable w.r.t. ODE, thus $A' = f_u + g_v < 0$ and $B' = f_u g_v - f_v g_u > 0$, and it implies $A > 0$. So we must have $B < 0$ (saddle type, and only one of $\mu_{1,k}$ or $\mu_{2,k}$ is positive.)

So we hope $B = \lambda^2(f_u g_v - f_v g_u) - k^2(df_u + g_v) + k^4 d < 0$ for some integer k. Since $f_u g_v - f_v g_u > 0$, then $df_u + g_v > 0$ otherwise $B > 0$ for all λ, k.

So we get necessary conditions of Turing instability:
$f_u g_v - f_v g_u > 0,\ f_u + g_v < 0,\ df_u + g_v > 0$

f_u and g_v must be of different sign, here we assume that $f_u < 0$ and $g_v > 0$. (so $0 < d < 1$)

Now from $\lambda^2(f_u g_v - f_v g_u) - k^2(df_u + g_v) + k^4 d < 0$, we solve d.

$$d < \frac{\lambda(g_v k^2 - \lambda DetJ)}{k^2(k^2 - \lambda f_u)} \text{ where } detJ = f_u g_v - f_v g_u.$$

So Turing instability will occur if

$$f_u g_v - f_v g_u > 0,\ f_u + g_v < 0,\ df_u + g_v > 0,\ f_u < 0,\ g_v > 0,$$

$$0 < \lambda < \frac{g_v k^2}{DetJ},\ \text{and } 0 < d < \frac{\lambda(g_v k^2 - \lambda DetJ)}{k^2(k^2 - \lambda f_u)}.$$

When these conditions are met, then k-th mode is unstable ($\cos(kx)$ is an eigenfunction). The band of unstable modes is given by $k^2 \in (a_1, a_2)$, where a_1 and a_2 are the roots of equation $\lambda^2(f_u g_v - f_v g_u) - a(df_u + g_v)\lambda + da^2 = 0$,

$$a = \frac{\lambda}{2d}\left[(df_u + g_v) \pm \sqrt{(df_u + g_v)^2 - 4dDetJ}\right]$$

.

An artificial example:

$$J = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -4 & 2 \end{pmatrix}$$

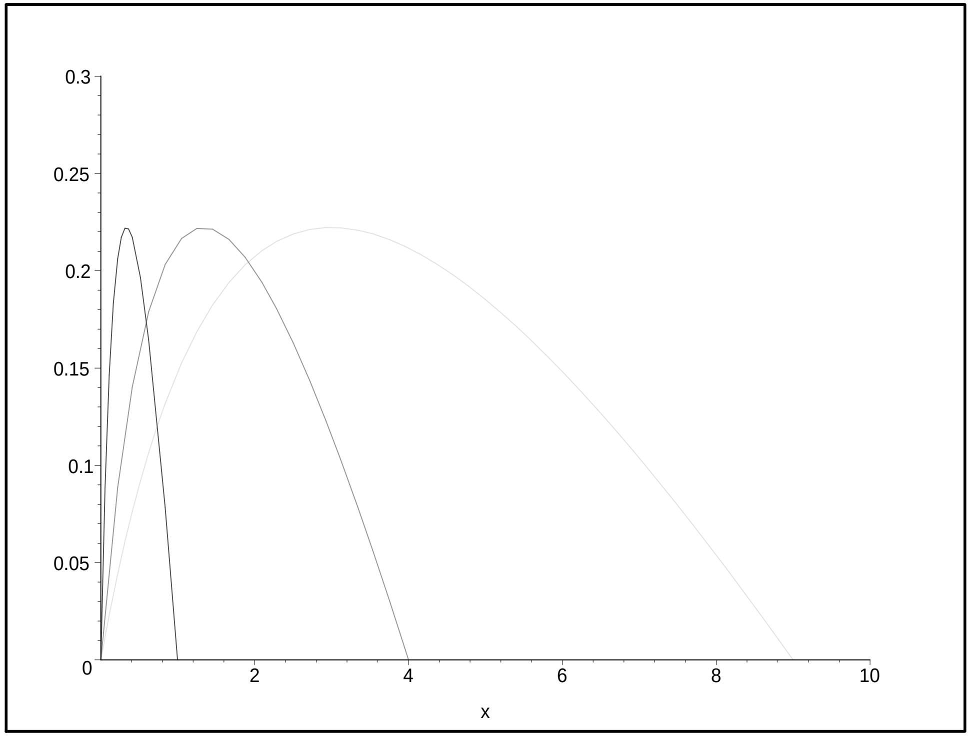
For ODE, it is stable, since $A' = f_u + g_v = -1 < 0$ and $B' = f_u g_v - f_v g_u = 2 > 0$.
We also have $f_u = -3 < 0$ and $g_v = 2 > 0$

Hence the k-th mode is unstable if

$$0 < \lambda < k^2, \;\; 0 < d < d_k(\lambda) \equiv \frac{2\lambda(k^2 - \lambda)}{k^2(k^2 + 3\lambda)}$$

$$d_1(\lambda) = \frac{2\lambda(1-\lambda)}{1+3\lambda}, \;\; d_2(\lambda) = \frac{2\lambda(4-\lambda)}{4(4+3\lambda)}, \;\; d_3(\lambda) = \frac{2\lambda(9-\lambda)}{9(9+3\lambda)}.$$

Parameter (λ, d) regions for Turing instability



Horizontal axis is λ, vertical one is d, and the graphs are $d_1(\lambda)$, $d_2(\lambda)$ and $d_3(\lambda)$.

For example, if $(\lambda, d) = (2, 0.1)$, then the band of unstable mode is $(2.54, 31.457)$, so k-mode ($k = 2, 3, 4, 5$) are unstable. From calculation of `Maple`, we find that the reaction-diffusion system has 4 positive eigenvalues. The largest eigenvalue is $\mu_{1,3} = 1.114$ with eigenvector $\mathbf{V} = (0.316, 1.273)$.

Hence near the equilibrium point, if we perturb the system, we have

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \approx \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + e^{1.114t} \cos(3x) \begin{pmatrix} 0.316 \\ 1.273 \end{pmatrix}$$

A spatial pattern with period $2\pi/3$ is generated by Turing instability. The characteristic wave number is $k = 3$, and the characteristic wave length is $2\pi/3$.

Turing bifurcations:

When the parameters (λ, d) change, then k-mode can turn from stable to unstable. For example, in our example, when $\lambda = 2$ and d decreases from 0.3 to 0, it cuts through $d_2(\lambda)$, $d_3(\lambda)$, $d_4(\lambda)$, and $d_5(\lambda)$. Each time it cuts a $d_k(\lambda)$, k-mode becomes unstable. Similar phenomenon occurs if we changes λ. Each time a k-mode becomes unstable, a curve of equilibrium solutions emerges from the constant equilibrium solution.

The properties of these curves are still unclear for most reaction-diffusion systems. People assume that the system will generate a spatial pattern with the most unstable mode of the constant solution, but this is not verified mathematically. However the Turing instability helps us when we do numerical simulation of the system.

Generalizations:

What we consider here is a problem in 1-d domain $(0, \pi)$, and without cross diffusion ($u_t = d_1 u_{xx} + d_2 v_{xx} + f(u, v)$) and convection u_x.

In 2-D square domain $S = (0, \pi) \times (0, \pi)$, the eigenfunction will be $\cos(mx)\cos(ny)$ for $m = 0, 1, \cdots$ and $n = 0, 1, \cdots$. Similar but more complicated calculations lead to Turing instability. If the eigenfunction is in form of $\cos(mx)$ or $\cos(ny)$ (without the other variable), the spatial pattern is <u>striped</u>; If the eigenfunction is in form of $\cos(mx)\cos(ny)$ (with both variables), the spatial pattern is <u>spotted</u>.