Paradiso, Purgatory, and Inferno model:

\[
\begin{pmatrix}
  x_1(n + 1) \\
  x_2(n + 1) \\
  x_3(n + 1)
\end{pmatrix}
= \begin{pmatrix}
  1.5(1 - d) & 0.5d & 0.1d \\
  0.75d & 1 - d & 0.1d \\
  0.75d & 0.5d & 0.2(1 - d)
\end{pmatrix}
\begin{pmatrix}
  x_1(n) \\
  x_2(n) \\
  x_3(n)
\end{pmatrix},
\]

\[
\begin{pmatrix}
  x_1(0) \\
  x_2(0) \\
  x_3(0)
\end{pmatrix}
= \begin{pmatrix}
  100 \\
  200 \\
  300
\end{pmatrix}.
\]

Shorter form: \( \bar{x}(n + 1) = A \cdot \bar{x}(n), \bar{x}(0) = (100; 200; 300). \)

The solution:

\( x(n) = A x(n - 1) = A^2 x(n - 2) = A^3 x(n - 3) = \cdots = A^n x(0) \)

(here \( A^n \) is the power of matrix \( A \), which is not easy to calculate, but easy for Matlab)
Observation from Matlab simulation:

\[ d = 0.2 \]
1. All population grow.
2. The total population grows exponentially. 
   \((\log(P) \text{ has a linear growth})\)
3. The fraction of each population in the total population tends to a constant

\[ d = 0.8 \]
1. All population decay.
2. The total population decays exponentially.
3. The fraction of each population in the total population tends to a constant
A stage structured model:
\[ x(n + 1) = \begin{pmatrix} 0 & 1.1 \\ 0.55 & 0.55 \end{pmatrix} x(n), \quad x(0) = (200; 0). \]

Observation: exponential growth, the numbers of juveniles and adults are almost same after a few time units.

For \( x(0) = \begin{pmatrix} 100 \\ 100 \end{pmatrix} \), then \( x(n) = A^n \begin{pmatrix} 100 \\ 100 \end{pmatrix} = 1.1^n \begin{pmatrix} 100 \\ 100 \end{pmatrix} \)

\[ A \begin{pmatrix} 100 \\ 100 \end{pmatrix} = 1.1 \begin{pmatrix} 100 \\ 100 \end{pmatrix}. \]

so the matrix power could be just the power of a scalar number. Linear algebra can explain all these. (Math 211)
A crash course of linear algebra

The numbers $\lambda$ satisfying $Ax = \lambda x$ are called eigenvalues. The corresponding $x(\neq 0)$ is called eigenvector associated with the eigenvalue. An $n \times n$ matrix has exactly $n$ eigenvalues (they could be same, but with different eigenvectors). If $v$ is an eigenvector, so is $cv$ ($c$ is a constant)

Example: 2 $\times$ 2 matrix

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}$
Eigenvalue and eigenvector of $2 \times 2$ matrix

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$

solve $(a - \lambda)(d - \lambda) - bc = 0$ (characteristic equation) for eigenvalues $\lambda_1, \lambda_2$

Example: Fibonacci model: $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Eigenvalues, eigenvectors for higher dimensional matrices: use Matlab to solve
Solution of $x(n + 1) = Ax(n)$ ($A$ is a $k \times k$ matrix):

$$x(n) = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2 + \cdots + c_k \lambda_k^n v_k = \sum_{i=1}^{k} c_i \lambda_i^n v_i$$

where $\lambda_i$ are eigenvalues, $v_i$ are eigenvectors, and $c_i$ are constants.

The eigenvalues can be ordered so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k|$, and $\lambda_1$ is called dominant eigenvalue.

$$x(n) \approx c_1 \lambda_1^n v_1 \text{ for large } n$$
Stability
If $|\lambda_1| < 1$, then $x(n) \to 0$ as $n \to \infty$ (extinction)

If $|\lambda_1| > 1$, then $x(n) \to \infty$ as $n \to \infty$ and $x(n) \approx c_1 \lambda_1^n v_1$

$\frac{1}{\lambda_1^n} x(n) \approx c_1 v_1$

Let $w_1 = \frac{v_1}{\text{sum}(v_1)}$, then $w_1$ is the stable state distribution of the model

Perron-Frobenius Theorem: If $A$ has non-negative entries and is power positive (there is a natural number $n$ such that $A^n$ has only positive entries), then the dominant eigenvalue of $A$ is positive, and the associated eigenvector is also positive.