Continuous time model and discrete time model:

continuous time model: $f(t)$ with time $0 \leq t < \infty$
discrete time model: $a_n$ with $n = 0, 1, 2, 3 \cdots$

continuous or discrete?
1. experimental data is discrete, but large amount of data is almost continuous
2. more mathematical tools are available for continuous model, computer can only analyze discrete model
3. continuous model can be discretized, and discrete model can be approximated by continuous one
4. continuous model needs calculus, discrete model needs only algebra (most of time)

continuous model:
differential equations (equation with derivatives)
Derivative of a function: \( f'(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} \)

If \( f(t) \) is the quantity or amount of certain organism, then \( f'(t) \) is the growth rate (rate of change) of that quantity. The derivative is used to check the change of quantity of the organism.

**Malthus Model**: Assumption: the reproduction rate is proportional to the size of the population

\[
\frac{dP}{dt} = kP, \quad k = \text{growth rate per capita}
\]

Solution: \( P(t) = P(0)e^{kt}, \) \( k > 0: \) growth, \( k < 0: \) decay

Proposed by Thomas Robert Malthus (1798) (British)

Validity: good for bacteria in an unlimited environment or computer virus, otherwise not reasonable due to the limitation of the resource.
**Logistic Model:**

Assumption: the reproduction rate is proportional to the size of the population when the population size is small, and the growth is negative when the size is large

\[ \frac{dP}{dt} = P \cdot g(P), \quad g(P) = \text{growth rate per capita} \]

Choose the simplest form which fulfils the assumptions:

\[ \frac{dP}{dt} = rP \left( 1 - \frac{P}{K} \right) \]

\( r = \text{maximum growth rate per capita, } K = \text{carrying capacity} \)

Proposed by Pierre Francois Verhulst (1838) (Belgian)

Examples: bacteria, yeast in a limited environment.
(laboratory experiments)
Qualitative analysis of Logistic equation:

\[ \frac{dP}{dt} = P \left(1 - \frac{P}{4}\right) \]

- \( P = 0 \) and \( P = 4 \) are equilibrium solutions (constant solutions.)

- When \( 0 < P(0) < 4 \), the solution is increasing, since
  \[ \frac{dP}{dt} = P \left(1 - \frac{P}{4}\right) > 0, \text{ and } \lim_{t \to \infty} P(t) = 4. \]

- When \( P(0) > 4 \), the solution is decreasing, since
  \[ \frac{dP}{dt} = P \left(1 - \frac{P}{4}\right) < 0, \text{ and } \lim_{t \to \infty} P(t) = 4. \]
Comparison of continuous and discrete models:

discrete Malthus: \( N_{n+1} = rN_n \), solution \( N_n = N_0 r^n \)

continuous Malthus \( P'(t) = rP(t) \), solution \( P(t) = P_0 e^{rt} \)

common behavior: exponential unlimited growth

Beverton-Holt model: \( N_{n+1} = \frac{R N_n}{1 + \frac{R-1}{K} N_n} \)

continuous logistic model: \( P' = rP \left(1 - \frac{P}{K}\right) \)

common behavior: \( S \)-shape bounded growth tend to the carrying capacity

discrete logistic model: \( x_{n+1} = rx_n(1 - x_n) \)

behavior: could be chaotic
Solving equations: (calculus, integral)

Method of separation of variables:

\[
\frac{dy}{dt} = f(t)g(y)
\]

\[
\frac{dy}{g(y)} = f(t)dt \Rightarrow \int \frac{dy}{g(y)} = \int f(t)dt
\]

Then solve \( y \) in term of \( t \) if possible.

Example:

(1) \( P' = \sin(t)P, \ P(0) = 3 \) (oscillatory population)

(2) \( x' = x^3 \) and \( x(0) = 1 \).
Solution of logistic equation: \( P' = rP \left(1 - \frac{P}{K}\right) \), \( P(0) = P_0 \)

use a change of variable \( Q(t) = 1/P(t) \)

\[
P(t) = \frac{KP_0}{(K - P_0)e^{-rt} + N_0}
\]

Solution of Beverton-Holt model: \( N_{n+1} = \frac{RN_n}{1 + \frac{R-1}{K}N_n} \)

use a change of variable \( y_n = 1/N_n \)

\[
N_n = \frac{KN_0}{(K - N_0)R^{-n} + N_0}
\]
Half life: \( P' = -kP, \ P(0) = P_0, \) the time \( T \) so that \( P(T) = P_0/2 \)

**Example** Polonium-210 has a half-life of 140 days.
(1) If a sample has a mass of 50 mg, find a formula for the mass after \( t \) days.
(2) Find the mass after 100 days.

**Example** *Determining an infusion rate* An asthmatic patient is given a continuous infusion of theophylline to relax and open the air passages in his lungs. The desired steady-state level of theophylline in the patients blood stream is 15 mg/L. The average half-life of theophylline is about 4 hours, and the patient has 5.6 liters of blood. Find the infusion required to maintain the desired steady-state level.