Regions containing a finite set of points

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Problem Given a finite set of complex numbers, determine the smallest regions with special geometrical shape containing them.

1 Convex hull

Let S be a finite set of complex numbers. Here we use an idea by Nam-Kiu Tsing to determine the convex hull of S as follows.

First detemine the extrema of the set $\{(z+\bar{z})/2 : z \in S\}$ and $\{(z-\bar{z})/(2i) : z \in S\}$, we get a rectangle touching k vertices of $\operatorname{conv} S$ with k. Assume $k > 2$ so that not all points are collinear.

Let $\gamma = (v_1 + \cdots + v_k)/k$ be the mean of the k vertices $v_1, \dots, v_k$. Replace S by $S - \gamma$. After this replacement, we may assume that $\operatorname{conv} S$ contains a k-side convex polygon which contains the origin.

Suppose the complex numbers $v_1, \dots, v_k$ are k vertices of $\operatorname{conv} S$ that have been identified such that

$$0 \leq \arg(v_1) \leq \cdots \leq \arg(v_k) \leq 2\pi.$$

Set $t_{k+1} = t_1$. For $j = 1, \dots, k$, either

(a) the line segment joining v_j and v_{j+1} is on the boundary $\operatorname{conv} S$, or

(b) the line joining v_j and v_{j+1} separates the origin and a vertex $v_{j'}$.

If (b) does not happen, then we are done. If (b) does happen, we can add $v_{j'}$ to the list of vertices, and do the classifiation of (a) and (b) again. If $\operatorname{conv} S$ has m vertices, this algorithm will stop after $m - k$ steps.

Note: To check whether (a) or (b) holds, we may let $\theta \in [0, 2\pi)$ such that the line $\mathcal{L}$ passing through $e^{i\theta}v_j$ and $e^{i\theta}v_{j+1}$ is a right supporting line of $\operatorname{conv}\{e^{i\theta}v_r : 1 \leq r \leq k\}$. Suppose

$$e^{i\theta}z + e^{-i\theta}\bar{z} \leq e^{i\theta}v_j + e^{-i\theta}\bar{v}_j$$

for all $z \in S$. Then condition (a) holds. Otherwise, condition (b) holds, and we can find $v_{j'}$ among those $z \in S$ such that $e^{i\theta}z$ has the maximum real part.

2 Smallest rectangle

Let S be a finite set of complex numbers. We want to determine the smallest rectangle containing S. Taking $P = \operatorname{conv} S$, we reduce the problem to the following.

Problem Determine the smallest rectangle containing a given n-side convex polygon P.

We can determine the rectangle in finite steps as shown by the following.

Theorem *For each side of the polygon P, construct a support line L of the polygon. Then there is a unique smallest rectangle with one of the side lying on L. There are at most n rectangles constructed in this way. The one with the minimum area is the desired one.*

Proof. We claim that if R is a rectangle containing P and none of the four sides of R contains an edge of P, then R is not optimal.

Note that if the hypothesis of the claim is true, then there are at most four vertices of P on the boundary of R. Moreover, none of these four vertices of P can be a vertex of R; otherwise, this vertex and one of the other vertices will be a boundary edge of P lying on the boundary of R.

Now, we may rotate $\mathbf{C}$ so that the sides of R are A, B, C, D so that A and C are horizontal with A above C, and that B and D are vertical with B on the right of D. Suppose the vertices a, b, c, d of P lies on the sides A, B, C, D or R. Let the line segment joining a and c be L, and let the line segment joining b and d be W. We may translate $\mathbf{C}$ so that the intersection of L and W is the origin. Let θ be the angle between L and the y axis, and ϕ be the angle between W and the x axis. Then the area of R is $lw\cos\theta\cos\phi$, where l and w are the lengths of L and W, respectively. If one can rotate $\mathbf{C}$ such that both θ and ϕ increase, then we can get a smaller rectangle containing P. Otherwise, both L and W must lie in (1) the first and third quadrants, or (2) the second and fourth quadrants. WOLOG, assume (1) holds. If we rotate $\mathbf{C}$ by an angle t in the clockwise direction, then the new angles between L and W to the vertical and horizontal axis will be $\cos(\theta+t)$ and $\cos(\phi-t)$, respectively. For sufficiently small t, we can find a new rectangle $R(t)$ containing P with area $f(t) = lw\cos(\theta+t)\cos(\phi-t)$. Since

$$f'(t) = -lw(\sin(\theta+t)\cos(\phi-t) - \cos(\theta+t)\sin(phi-t)) = -lw\sin(\theta+\phi) < 0,$$

we can rotate $\mathbf{C}$ and obtain a new rectangle containing P with sides parallel to the axises and a smaller area. Thus, our claim is proved and the result follows. $\square$

3 Further questions

(a) Determine the smallest ellipse containing S.
Every 5 vertices of $\operatorname{conv} S$ determine an ellipse. Can we do better?

(b) How about extending the result to $\mathbf{R}^3$ or higher dimensions?

(c) How about using a (regular) n-side polygon to contain a given convex polygon (convex set)?